Theoretical Question 1

When will the Moon become a Synchronous Satellite?

The period of rotation of the Moon about its axis is currently the same as its period of revolution about the Earth so that the same side of the Moon always faces the Earth. The equality of these two periods presumably came about because of actions of tidal forces over the long history of the Earth-Moon system.

However, the period of rotation of the Earth about its axis is currently shorter than the period of revolution of the Moon. As a result, lunar tidal forces continue to act in a way that tends to slow down the rotational speed of the Earth and drive the Moon itself further away from the Earth.

In this question, we are interested in obtaining an estimate of how much more time it will take for the rotational period of the Earth to become equal to the period of revolution of the Moon. The Moon will then become a synchronous satellite, appearing as a fixed object in the sky and visible only to those observers on the side of the Earth facing the Moon. We also want to find out how long it will take for the Earth to complete one rotation when the said two periods are equal.

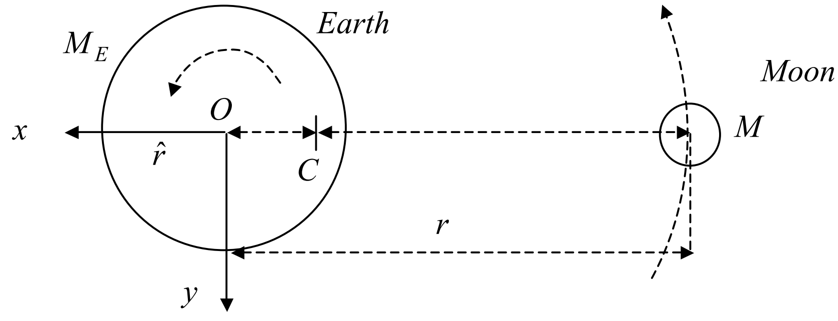
Two right-handed rectangular coordinate systems are adopted as reference frames. The third coordinate axes of these two systems are parallel to each other and normal to the orbital plane of the Moon.

(I)The first frame, called the *CM* frame, is an inertial frame with its origin located at the center of mass C of the Earth-Moon system.

(II)The second frame, called the *xyz* frame, has its origin fixed at the center O of the Earth. Its z-axis coincides with the axis of rotation of the Earth. Its x-axis is along the line connecting the centers of the Moon and the Earth, and points in the direction of the unit vector $\hat{r}$ as shown in Fig.1a. The Moon remains always on the negative x-axis in this frame.

Note that distances in Fig.1a are not drawn to scale. The curved arrows show the directions of the Earth's rotation and the Moon's revolution. The Earth-Moon distance is denoted by r.

Fig. 1a



The following data are given:

(a) At present, the distance between the Moon and the Earth is $r_0 = 3.85\times10^8$ m and increases at a rate of 0.038 m per year.

(b) The period of revolution of the Moon is currently T_M = 27.322 days.

(c) The mass of the Moon is $M = 7.35\times10^{22}$ kg.

(d) The radius of the Moon is $R_M = 1.74\times10^6$ m.

(e) The period of rotation of the Earth is currently T_E = 23.933 hours.

(f) The mass of the Earth is $M_E = 5.97\times10^{24}$ kg.

(g) The radius of the Earth is $R_E = 6.37\times10^6\,m$.

(h) The universal gravitational constant is $G = 6.67259\times10^{-11}\,N\cdot m^2/kg^2$.

The following assumptions may be made when answering questions:

(i) The Earth-Moon system is isolated from the rest of the universe.

(ii) The orbit of the Moon about the Earth is circular.

(iii) The axis of rotation of the Earth is perpendicular to the orbital plane of the Moon.

(iv) If the Moon is absent and the Earth does not rotate, then the mass distribution of the Earth is spherically symmetric and the radius of the Earth is R_E.

(v) For the Earth or the Moon, the moment of inertia I about any axis passing through its center is that of a uniform sphere of mass M and radius R, i.e. $I = \frac{2}{5}MR^2$.

(vi) The water around the Earth is stationary in the *xyz* frame.

Answer the following questions:

(1) With respect to the center of mass C, what is the current value of the total angular momentum L of the Earth-Moon system?

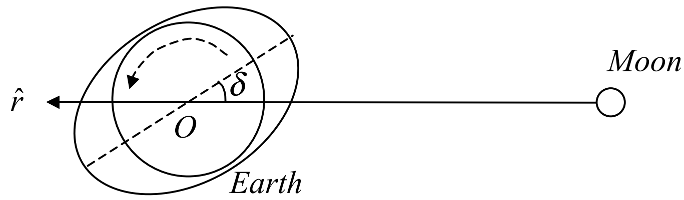
(2) When the period of rotation of the Earth and the period of revolution of the Moon become equal, what is the duration of one rotation of the Earth? Denote the answer as T and express it in units of the present day. Only an approximate solution is required so that iterative methods may be used.

(3) Consider the Earth to be a rotating solid sphere covered with a surface layer of water and assume that, as the Moon moves around the Earth, the water layer is stationary in the *xyz*-frame. In one model, frictional forces between the rotating solid sphere and the water layer are taken into account. The faster spinning solid Earth is assumed to drag lunar tides along so that the line connecting the tidal bulges is at an angle δ with the *x*-axis, as shown in Fig.1b. Consequently, lunar tidal forces acting on the Earth will exert a torque Γ about O to slow down the rotation of the Earth.

The angle δ is assumed to be constant and independent of the Earth-Moon distance r until it vanishes when the Moon's revolution is synchronous with the Earth's rotation so that frictional forces no longer exist. The torque Γ therefore scales with the Earth-Moon distance and is proportional to $\frac{1}{r^6}$.

According to this model, when will the rotation of the Earth and the revolution of the Moon have the same period? Denote the answer as t_f and express it in units of the present year.

Fig.1b



The following mathematical formulae may be useful when answering questions:

(M1) For $0 \le s < r$ and $x = s\cos\theta$:

$$\frac{1}{\sqrt{r^2+s^2+2rx}} \approx (\frac{1}{r} - \frac{x}{r^2} + \frac{3x^2 - s^2}{2r^3} + \cdots)$$

(M2) If $a \neq 0$ and $\frac{d\omega}{dt} = b\omega^{1-a}$, then $\omega^a(t') - \omega^a(t) = (t'-t)ab$

Theoretical Question 2

Motion of an Electric Dipole in a Magnetic Field

In the presence of a constant and uniform magnetic field $\vec{B}$, the translational motion of a system of electric charges is coupled to its rotational motion. As a result, the conservation laws for the *momentum* and the *component of the angular momentum* along the direction of $\vec{B}$ are modified from the usual form. This is illustrated in this problem by considering the motion of an ***electric dipole*** made of two particles of equal mass m and carrying *charges* q and $-q$ respectively ($q > 0$). The two particles are connected by a rigid insulating rod of *length* ℓ, the mass of which can be neglected. Let $\vec{r}_1$ be the *position vector* of the particle with charge q, $\vec{r}_2$ that of the other particle and $\vec{\ell} = \vec{r}_1 - \vec{r}_2$. Denote by $\vec{\omega}$ the *angular velocity* of the rotation around the center of mass of the dipole. Denote by $\vec{r}_{CM}$ and $\vec{v}_{CM}$ the *position* and the *velocity* vectors of the *center of mass* respectively. *Relativistic effects and effects of electromagnetic radiation can be neglected.*

Note that the *magnetic force* acting on a particle of charge q and velocity $\vec{v}$ is $q\vec{v} \times \vec{B}$, where the cross product of two vectors $\vec{A}_1 \times \vec{A}_2$ is defined, in terms of the x, y, z, components of the vectors, by

$$(\vec{A}_1 \times \vec{A}_2)_x = (\vec{A}_1)_y (\vec{A}_2)_z - (\vec{A}_1)_z (\vec{A}_2)_y$$
$$(\vec{A}_1 \times \vec{A}_2)_y = (\vec{A}_1)_z (\vec{A}_2)_x - (\vec{A}_1)_x (\vec{A}_2)_z$$
$$(\vec{A}_1 \times \vec{A}_2)_z = (\vec{A}_1)_x (\vec{A}_2)_y - (\vec{A}_1)_y (\vec{A}_2)_x.$$

(1) **Conservation Laws**

(a) Write down the *equations of motion* for the *center of mass* of the dipole and for the *rotation around* the center of mass by computing the total force and the total torque with respect to the center of mass acting on the dipole.

(b) From the equation of motion for the center of mass, obtain the ***modified*** *form* of the conservation law for the *total momentum*. Denote the corresponding modified conserved quantity by $\vec{P}$. Write down an *expression* in terms of $\vec{v}_{CM}$ and $\vec{\omega}$ for the conserved *energy E*.

(c) The angular momentum consists of two parts. One part is due to the motion of the center of mass and the other is due to rotation around the center of mass. From the modified form of the conservation law for the total momentum and the equation of motion of the rotation around the center of mass, prove that the quantity J as defined by

$$J = (\vec{r}_{CM} \times \vec{P} + I\vec{\omega}) \cdot \hat{B}$$

is conserved.

Note that

$$\vec{A}_1 \times \vec{A}_2 = -\vec{A}_2 \times \vec{A}_1$$

$$\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3) = (\vec{A}_1 \times \vec{A}_2) \cdot \vec{A}_3$$

$$\vec{A}_1 \times (\vec{A}_2 \times \vec{A}_3) = (\vec{A}_1 \cdot \vec{A}_3)\vec{A}_2 - (\vec{A}_1 \cdot \vec{A}_2)\vec{A}_3$$

for any three vectors $\vec{A}_1$, $\vec{A}_2$ and $\vec{A}_3$. Repeated application of the above first two formulas may be useful in deriving the conservation law in question.

In the following, let $\vec{B}$ be in the z-direction.

(2) **Motion in a Plane Perpendicular to** $\vec{B}$

Suppose initially the center of mass of the *dipole* is *at rest* at the origin, $\vec{\ell}$ points in the x-direction and the *initial angular velocity* of the dipole is $\omega_0 \hat{z}$ ($\hat{z}$ is the unit vector in the z-direction).

(a) If the magnitude of ω_0 is smaller than a *critical value* ω_c, the dipole will not make *a full turn* with respect to its center of mass. Find ω_c.

(b) For a general $\omega_0 > 0$, what is the *maximum distance* d_m in the x-direction that the center of mass can reach?

(c) What is the *tension* on the rod? Express it as a function of the angular velocity ω.

Theoretical Question 3

Thermal Vibrations of Surface Atoms

This question considers the thermal vibrations of surface atoms in an elemental metallic crystal with a face-centered cubic (*fcc*) lattice structure. The unit cubic cell of an *fcc* lattice consists of one atom at each corner and one atom at the center of each face of the cubic cell, as shown in Fig. 3a. For the crystal under consideration, we use (a, 0, 0), (0, a, 0) and (0, 0, a) to represent the locations of the three atoms on the x, y and z axes of its cell. The lattice constant a is equal to 3.92 Å (i.e..the length of each side of the cube is 3.92 Å).

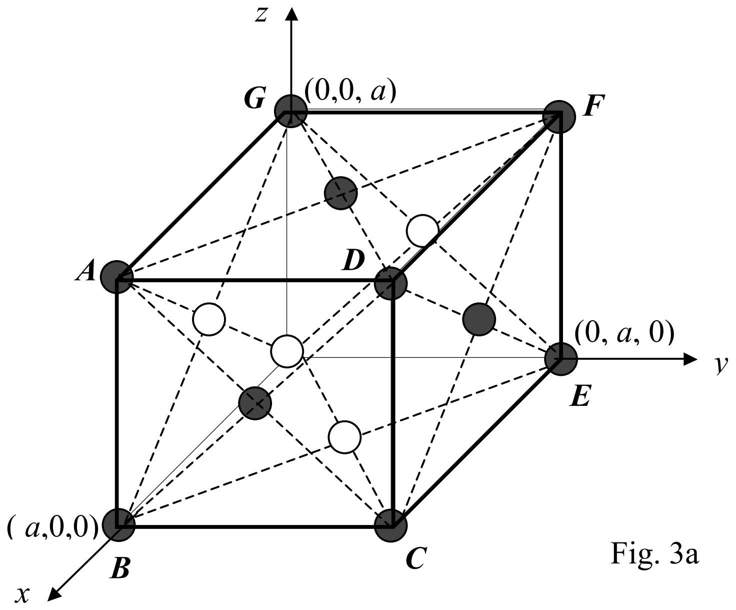


Fig. 3a

(1) The crystal is cut in such a way that the plane containing ABCD becomes a boundary surface and is chosen for doing low-energy electron diffraction experiments. A collimated beam of electrons with kinetic energy of 64.0 eV is incident on this surface plane at an incident angle ϕ_0 of 15.0° . Note that ϕ_0 is the angle between the incident electron beam and the normal of the surface plane. The plane containing $\overline{AC}$ and the normal of the surface plane is the plane of incidence. For simplicity, we assume that all incident electrons are back scattered only by the surface atoms on the topmost layer.

(a) What is the wavelength of the matter waves of the incident electrons?

(b) If a detector is set up to detect electrons that do not leave the plane of incidence after being diffracted, at what angles with the normal of the surface will these diffracted electrons be observable?

(2) Assume that the thermal vibrational motions of the surface atoms are simple

harmonic. The amplitude of vibration increases as the temperature rises. Low-energy electron diffraction provides a way to measure the average amplitude of vibration. The intensity I of the diffracted beam is proportional to the number of scattered electrons per second. The relation between the intensity I and the displacement $\vec{u}$ (t) of the surface atoms is given by

$$I = I_0 \ \exp \ \{ - < [(\vec{K}' - \vec{K}) \cdot \vec{u}]^2 > \} \qquad (1)$$

In Eq.(1), I and I_0 are the intensities at temperature T and absolute zero, respectively. $\vec{K}$ and $\vec{K}'$ are wave vectors of incident electron and diffracted electron, respectively. The angle brackets < > is used to denote average over time. Note that the relation between the wave vector $\vec{K}$ and the momentum $\vec{p}$ of a particle is $\vec{K} = 2\pi \ \vec{p} \,/\, h$, where h is the Planck constant.

To measure vibration amplitudes of surface atoms of a metallic crystal, a collimated electron beam with kinetic energy of 64.0 eV is incident on a crystal surface at an incident angle of 15.0°. The detector is set up for measuring specularly reflected electrons. Only elastically scattered electrons are detected. A plot of $ln \ (I / I_0)$ versus temperature T is shown in Fig. 3b.

Assume the total energy of an atom vibrating in the direction of the surface normal $\hat{x}$ is given by $k_B T$, where k_B is the Boltzmann constant.

(a) Calculate the *frequency of vibration* in the direction of the surface normal for the surface atoms.

(b) Calculate the *root-mean-square displacement*, i. e. the value of $(< u_x^2 >)^{1/2}$, in the direction of the surface normal for the surface atoms at *300 K*.

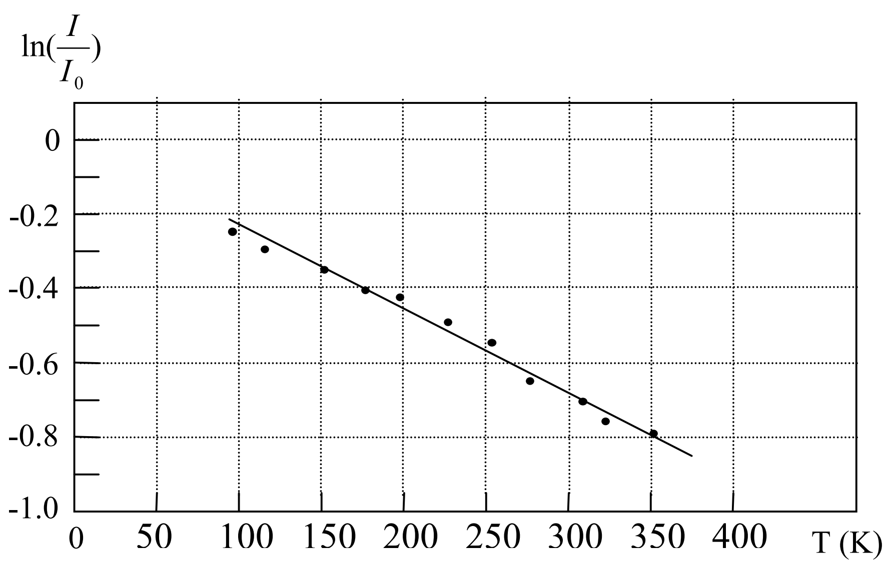


Fig. 3b

The following data are given:

Atomic weight of the metal $M = 195.1$

Boltzmann constant $k_B = 1.38\times10^{-23}$ *J/K*

Mass of electron = 9.11×10^{-31} *kg*

Charge of electron = 1.60×10^{-19} *C*

Planck constant $h = 6.63 \times10^{-34}$ *J-s*