[Solution] *Theoretical Question 1*

When will the Moon become a Synchronous Satellite?

(1) The total orbital angular momentum $\vec{L} = L\,\hat{z}$ of the Earth-Moon system with respect to C can be calculated as follows.

Since all angular momenta are along the z-direction, only the z-component of each angular momentum will be calculated. The distance between the center of mass C and the center of the Earth O is

$$r_{CM} = \frac{Mr_0}{M+M_E} = \frac{3.85\times10^8}{1+(597/7.35)} = 4.68\times10^6\,m = 0.735\,R_E$$

The *angular speed* of the *Moon*'s *revolution* is

$$\omega_0 = \frac{2\pi}{27.322\times86400} = 2.6617\times10^{-6}\ rad/s \qquad (1a)$$

The *orbital* angular momentum of the Moon about C is

$$\ell_M = M(r_0 - r_{CM})^2\omega_0$$
$$= 7.35\times(385-4.68)^2\times2.6617\times10^{28} = 2.83\times10^{34}\,kg\cdot m^2/s$$

The *angular speed* of the Moon's *spinning* or *rotational* motion is

$$\Omega_M = \omega_0 = 2.6617\times10^{-6}\ rad/s$$

The *spin angular momentum* of the Moon is

$$S_M = \frac{2}{5}MR_M^2\Omega_M = \frac{2}{5}\times7.35\times(1.74)^2\times2.6617\times10^{28}$$
$$= 2.37\times10^{29}\,kg\cdot m^2/s = 8.40\times10^{-6}\,\ell_M$$

This is much smaller than the Moon's orbital angular momentum and can therefore be neglected.

The *orbital* angular momentum of the *Earth* about C is

$$\ell_E = M_E r_{CM}^2\omega_0 = \frac{M}{M_E}\ell_M$$
$$= \frac{7.35}{597}\times2.83\times10^{34} = 3.48\times10^{32}\ kg\cdot m^2/s$$

The angular speed of the Earth's spinning motion is

$$\Omega_E = \frac{2\pi}{23.933\times3600} = 7.2926\times10^{-5}\ rad/s$$

The *moment of inertia* of the Earth about its axis of rotation is

$$I = \frac{2}{5}M_E R_E^2 = 0.4\times5.97\times(6.37)^2\times10^{36} = 9.69\times10^{37}\ km\cdot m^2/s \qquad (1b)$$

The *spin* angular momentum of the Earth is

$$S_E = \frac{2}{5}M_E R_E^2\Omega_E = 7.07\times10^{33}\,kg\cdot m^2/s = 20.3\ \ \ell_E$$

*[**Solution**] (continued)* *Theoretical Question 1*

When will the Moon become a Synchronous Satellite?

Thus, the *total angular momentum* of the Earth-Moon system L is given by

$$\begin{aligned} L &= (\ell_M + \ell_E + S_E + S_M) \\ &= (2.83 + 0.0348 + 0.707 + 0.0000237) \times 10^{34} \\ &= 3.57 \times 10^{34}\ kg \cdot m^2 / s \end{aligned} \tag{2}$$

Note that $L \approx (\ell_M + \ell_E + S_E)$.

(2) According to Newton's form for *Kepler's third law* of planetary motions, the angular speed ω of the revolution of the Moon about the Earth is related to the Earth-Moon distance r by

$$\omega^2 r^3 = G(M_E + M) \tag{3}$$

Therefore, the *orbital angular momentum* of the Earth-Moon system with respect to C is

$$\ell = \left(\frac{M_E M}{M + M_E}\right) r^2 \omega = M M_E \left(\frac{G^2}{\omega(M + M_E)}\right)^{1/3} \tag{4}$$

(Note: $\ell_M = M\left(\frac{M_E r}{M + M_E}\right)^2 \omega$, $\ell_E = M_E\left(\frac{M r}{M + M_E}\right)^2 \omega$ so that $\ell = \ell_E + \ell_M$.)

When the angular speed of the Earth's rotation is equal to the angular speed ω of the orbiting Moon, the total angular momentum of the Earth-Moon system is, with the *spin angular momentum of the Moon neglected*, given by

$$\begin{aligned} (\ell_M + \ell_E + S_E) &= M M_E \left\{\frac{G^2}{(M + M_E)\omega}\right\}^{1/3} + \frac{2}{5} M_E R_E^2 \omega \\ &= 7.35 \times 5.97 \times \left\{\frac{66.726 \times 66.726}{(5.97 + 0.0735)\omega}\right\}^{1/3} \times 10^{30} + 9.69 \times 10^{37} \omega \\ &= 3.96 \times 10^{32} \omega^{-1/3} + 9.69 \times 10^{37} \omega = 3.57 \times 10^{34} \end{aligned} \tag{5a}$$

The last equality follows from conservation of total angular momentum and Eq.(2). For an initial estimate of ω, the spin angular momentum of the Earth may be neglected in Eq.(5a) to give

$$\omega \approx \omega_1 = \left(\frac{3.96}{357}\right)^3 = 1.36 \times 10^{-6}\ \ rad/s \quad \text{(first iteration)}$$

An improved estimate may be obtained by using the above estimated value ω_1 to compute the spin angular momentum of the Earth and use Eq.(5a) again to solve for ω. The result is

*[**Solution**] (continued) Theoretical Question 1*

When will the Moon become a Synchronous Satellite?

$$\omega \approx \omega_f = (\frac{3.96}{358})^3 = 1.35\times10^{-6} \ \ rad/s \quad \text{(second iteration)} \qquad (5b)$$

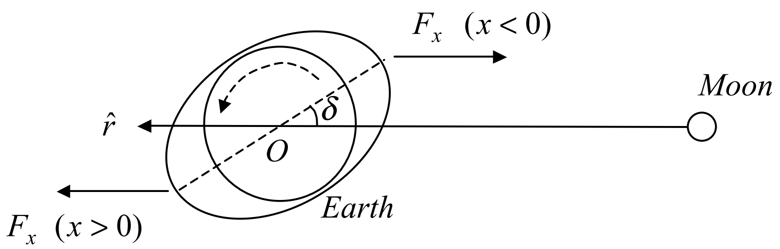
Further iterations of the same procedure lead to the same value just given. Thus, the *period of rotation of the Earth* will be

$$T_f = \frac{2\pi}{\omega_f} = \frac{6.2832}{1.35\times10^{-6}\times86400} = 53.9 \ days$$

(3) Since the *total torque* Γ is proportional to $1/r^6$, we conclude

$$r^6\,\Gamma = \text{constant} \qquad (6)$$

Fig.1e



Let the current values of r and Γ be, respectively, r_0 and Γ_0. From Eq.(6), we then have

$$\Gamma = (\frac{r_0}{r})^6\,\Gamma_0 \qquad (7)$$

The torque Γ is equal to the *rate of change* of spin angular momentum $I\,\Omega$ of the *Earth* so that

$$I\frac{d\Omega}{dt} = \Gamma \qquad (8)$$

By Newton's law of action and reaction or by the law of conservation of the total angular momentum, $-\Gamma$ is equal to the *rate of change* of the total orbital angular momentum ℓ of the *Earth-Moon system* so that

$$\frac{d\ell}{dt} = -\Gamma \qquad (9)$$

But according to Eq.(3), we have

$$\omega^2 r^3 = G(M_E + M)$$

and Eq.(4) may be written as

$$\ell = (\frac{MM_E}{M_E + M})\omega r^2 = MM_E(\frac{G}{M_E + M})^{1/2} r^{1/2}$$
$$= MM_E(\frac{G^2}{M_E + M})^{1/3} \omega^{-1/3} \tag{10}$$

This implies

$$\frac{d\ell}{dt} = MM_E(\frac{G}{M_E + M})^{1/2} \frac{1}{2r^{1/2}} \frac{dr}{dt}$$
$$= -\frac{1}{3} MM_E(\frac{G^2}{M_E + M})^{1/3} \frac{1}{\omega^{4/3}} \frac{d\omega}{dt} = -\Gamma \tag{11a}$$

The value of Γ_0 can be determined from Eq. (11a) as follows:

$$-\Gamma_0 = (\frac{d\ell}{dt})_0 = \frac{1}{2} MM_E \sqrt{\frac{G}{(M_E + M)r_0}} (\frac{dr}{dt})_0$$
$$= \frac{1}{2} \times 7.35 \times 5.97 \times 10^{46} \cdot \sqrt{\frac{66.7 \times 10^{-44}}{(0.0735 + 5.97) \times (3.85)}} \cdot \frac{3.8 \times 10^{-8}}{3.65 \times 8.64} \tag{13}$$
$$= 4.5 \times 10^{16} \ N \cdot m$$

Starting with Eq.(11a) in the following form

$$\frac{d\ell}{dt} = -\frac{1}{3} MM_E(\frac{G^2}{(M_E + M)})^{1/3} \frac{1}{\omega^{4/3}} \frac{d\omega}{dt} = -(\frac{r_0}{r})^6 \Gamma_0$$

we may use Eq.(3) to express r in terms of ω and obtain the following equations:

$$\frac{1}{3} MM_E(\frac{G^2}{(M_E + M)})^{1/3} (\frac{d\omega}{dt}) = \frac{(r_0)^6 \Gamma_0}{\{G(M_E + M)\}^2} \omega^{16/3}$$

*[**Solution**] (continued) Theoretical Question 1*

When will the Moon become a Synchronous Satellite?

$$\frac{d\omega}{dt} = [\frac{3(r_0)^6 \Gamma_0}{GM_E M\{G(M_E + M)\}^{5/3}}]\omega^{16/3} = b\omega^{16/3}$$

where the constant b stands for the expression in the square brackets. The last equation leads to the solution

$$(\omega_f)^{-13/3} - (\omega_0)^{-13/3} = \frac{-13b}{3}(t_f - 0)$$

where t_f is the length of time needed for the angular speed of the rotation of the Earth to be equal to that of the Moon's revolution about the Earth.

Using the values of ω_f and ω_0 obtained in Eqs.(1a) and (5b) and the value of Γ_0 in Eq.(13), we have

$$\frac{-3}{13b} = \frac{GM_E M\{G(M_E + M)\}^{5/3}}{13(r_0)^6(-\Gamma_0)} = 3.4 \times 10^{-8}$$

$$t_f = \frac{-3}{13b}(\omega_f^{-13/3} - \omega_0^{-13/3}) = 3.4 \times \{(1.35)^{-13/3} - (2.6617)^{-13/3}\} \times 10^{18}$$
$$= 3.4 \times 10^{18} \times (0.254 - 0.014376)$$
$$= 8.1 \times 10^{17} \sec onds$$
$$= 2.6 \times 10^{10} \, years$$

[Solution] *Theoretical Question 2*

Motion of an Electric Dipole in a Magnetic Field

(1) Conservation Laws

(1a) $\vec{r}_{CM} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$, $\vec{v}_{CM} = \frac{1}{2}(\vec{v}_1 + \vec{v}_2)$, $\vec{\ell} = \vec{r}_1 - \vec{r}_2$, $\vec{u} = \dot{\vec{\ell}} = \vec{v}_1 - \vec{v}_2$

Total force $\vec{F}$ on the dipole is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = q(\vec{E} + \vec{v}_1 \times \vec{B}) + (-q)(\vec{E} + \vec{v}_2 \times \vec{B}) = q(\vec{v}_1 - \vec{v}_2) \times \vec{B}$$
$$= q\dot{\vec{\ell}} \times \vec{B}$$

so that

$$M\dot{\vec{v}}_{CM} = q\dot{\vec{\ell}} \times \vec{B} \qquad (M = 2m) \tag{1}$$

Computing the torque for rotation around the center of mass, we obtain

$$\begin{aligned} I\dot{\vec{\omega}} &= (\frac{\vec{\ell}}{2}) \times (q\vec{v}_1 \times \vec{B}) + (\frac{-\vec{\ell}}{2}) \times (-q\vec{v}_2 \times \vec{B}) \\ &= q\vec{\ell} \times (\vec{v}_{CM} \times \vec{B}) \end{aligned} \tag{2}$$

where

$$I = \frac{1}{2} m\ell^2 \tag{3}$$

(1b) From *eq*.(1), we obtain the conservation law for the momentum:

$$\dot{\vec{P}} = 0, \quad \vec{P} = M\vec{v}_{CM} - q\vec{\ell} \times \vec{B} \tag{4}$$

From *eq*.(1) and *eq*.(2), one obtains the conservation law for the energy.

$$\dot{E} = 0, \quad E = \frac{1}{2} Mv_{CM}^2 + \frac{1}{2} I\omega^2 \tag{5}$$

(1c) Using *eq*.(4) and *eq*.(2),

$$\begin{aligned} \frac{d}{dt}(\vec{r}_{CM} \times \vec{P}) \cdot \hat{B} &= (\vec{v}_{CM} \times \vec{P}) \cdot \hat{B} = -q\vec{v}_{CM} \times (\vec{\ell} \times \vec{B}) \cdot \hat{B} \\ &= q(\vec{\ell} \times \vec{B}) \times \vec{v}_{CM} \cdot \hat{B} = q(\vec{\ell} \times \vec{B}) \cdot (\vec{v}_{CM} \times \hat{B}) \\ &= q\vec{\ell} \cdot (\vec{B} \times (\vec{v}_{CM} \times \hat{B})) = -q\vec{\ell} \cdot ((\vec{v}_{CM} \times \vec{B}) \times \hat{B}) \\ &= -q\vec{\ell} \times (\vec{v}_{CM} \times \vec{B}) \cdot \hat{B} \\ &= -I\dot{\vec{\omega}} \cdot \hat{B} \end{aligned}$$

*[**Solution**] (continued)* *Theoretical Question 2*

Motion of an Electric Dipole in a Magnetic Field

we obtain the conservation law

$$\dot{J}=0 \qquad J=(\vec{r}_{CM}\times\vec{P}+I\vec{\omega})\cdot\hat{B} \tag{6}$$

for the component of the angular momentum along the direction of $\vec{B}$.

(2) Motion in a Plane Perpendicular to $\vec{B}$

(2a) Write

$$\vec{\ell}=\ell\{\cos\varphi(t)\,\hat{x}+\sin\varphi(t)\,\hat{y}\},\quad \varphi(0)=0,\ \dot{\varphi}(0)=\omega_0 \tag{7}$$

Note that

$$\vec{\omega}=\dot{\varphi}\,\hat{z} \tag{8}$$

From *eq*.(4), we have

$$M\vec{v}_{CM}=\vec{P}+q\ell B\,(\sin\varphi\,\hat{x}-\cos\varphi\,\hat{y}) \tag{9}$$

At $t = 0$, we have $v_{CM} = 0$, $\varphi = 0$ so that

$$\vec{P}=q\ell B\,\hat{y} \tag{10}$$

Hence from *eqs*.(9) and (10) we have

$$\dot{x}_{CM}=\left(\frac{q\ell B}{M}\right)\sin\varphi,\quad \dot{y}_{CM}=\left(\frac{q\ell B}{M}\right)(1-\cos\varphi) \tag{11}$$

From conservation of energy, i.e. *Eq*.(5), we have

$$\frac{1}{2}I\dot{\varphi}^2+\frac{(q\ell B)^2}{M}(1-\cos\varphi)=\frac{1}{2}I\omega_0^2$$

$$\therefore\quad \dot{\varphi}^2+\frac{1}{2}\omega_c^2(1-\cos\varphi)=\omega_0^2 \tag{12}$$

where

$$\omega_c^2=\frac{4(q\ell B)^2}{MI}=\left(\frac{2qB}{m}\right)^2 \tag{13}$$

In order to make a full turn, $\dot{\varphi}$ can not become zero so that

*[**Solution**] (continued) Theoretical Question 2*

Motion of an Electric Dipole in a Magnetic Field

$$\omega_0^2 > \omega_c^2 \;\Rightarrow\; |\omega_0| > \omega_c = \frac{2qB}{m} \qquad (14)$$

(2b) From *Eq*.(6), we have

$$x_{CM} P + I\omega = J \qquad (15)$$

where P is the magnitude of $\vec{P}$.

At $t = 0$, we have $J = I\omega_0$ so that

$$x_{CM} P + I\omega = I\omega_0 \qquad (16)$$

From *eq*.(12), one can see that $\omega_0^2 \geq \omega^2$ so that $x_{CM} \geq 0$. Thus x_{CM} reaches a maximum d_m when ω takes its minimum value.

When $\omega_0 < \omega_c$, the minimum value of ω is $-\omega_0$ so that

$$d_m = \frac{2I}{P}\omega_0 = \left(\frac{m\omega_0}{qB}\right)\ell, \quad \omega_0 < \omega_c \qquad (17)$$

When $\omega_0 > \omega_c$, the minimum value of ω is $\sqrt{\omega_0^2 - \omega_c^2}$ so that

$$d_m = \left(\frac{I}{P}\right)\left(\omega_0 - \sqrt{\omega_0^2 - \omega_c^2}\right) = \frac{m}{2qB}\left(\omega_0 - \sqrt{\omega_0^2 - \omega_c^2}\right)\ell, \quad \omega_0 > \omega_c \qquad (18)$$

When $\omega_0 = \omega_c$, $\omega^2 = \frac{1}{2}\omega_c^2(1+\cos\varphi) = \omega_c^2 \cos^2\frac{\phi}{2}$

$$\therefore \quad \dot{\varphi} = \omega_c \cos\frac{\phi}{2}$$

When φ is close to π, let $\varphi = \pi - 2\varepsilon$ then

$$\dot{\varepsilon} = -\frac{1}{2}\omega_c \sin\varepsilon \approx -\frac{1}{2}\omega_c\varepsilon$$

$$\therefore \quad \varepsilon \sim e^{-\omega_c t/2}$$

so that it will take $t \to \infty$ for $\varepsilon \to 0$, i.e. for φ to reach π. Hence

$$d_m = \left(\frac{I}{P}\right)\omega_c = \left(\frac{m\omega_c}{2qB}\right)\ell, \quad \omega_0 = \omega_c \qquad (19)$$

(2c) Tension on the rod comes from three sources:

*[**Solution**] (continued)* *Theoretical Question 2*

Motion of an Electric Dipole in a Magnetic Field

(i) Coulomb force $= \dfrac{1}{4\pi\varepsilon_0}\dfrac{q^2}{\ell^2}$ (20)

Positive value means compression on the rod.

(ii) Centrifugal force due to rotation of the rod $= -\dfrac{1}{2}m\omega^2\ell$ (21)

(iii) Magnetic force on the particles due to the motion of the center of the mass

$$= q\vec{v}_{CM}\times\vec{B}\cdot(-\hat{\ell}) = q\vec{v}_{CM}\cdot\hat{\ell}\times\vec{B}$$

Taking the square of both sides of *eq*.(4) and using the initial condition for the value of P^2, we obtain

$$\frac{1}{2}Mv_{CM}^2 = q\ell\vec{v}_{CM}\cdot\hat{\ell}\times\vec{B} = \frac{1}{2}I(\omega_0^2-\omega^2) \tag{22}$$

Combining the three forces, we have

$$\textit{tension on the rod} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{\ell^2} - \frac{1}{2}m\ell\omega^2 + \frac{1}{4}m\ell(\omega_0^2-\omega^2) \tag{23}$$

A positive value means compression on the rod.

[Solution] *Theoretical Question 3*

Thermal Vibration of Surface Atoms

(1) (a) The wavelength of the incident electron is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2meV}}$$
$$= \frac{6.63\times10^{-34}}{\sqrt{2\times9.11\times10^{-31}\times1.60\times10^{-19}\times64.0}}$$
$$= 1.53\times10^{-10}\,m = 1.53\,\mathring{A}$$

(b) Consider the interference between the atomic rows on the surface as shown in Fig. 3c.

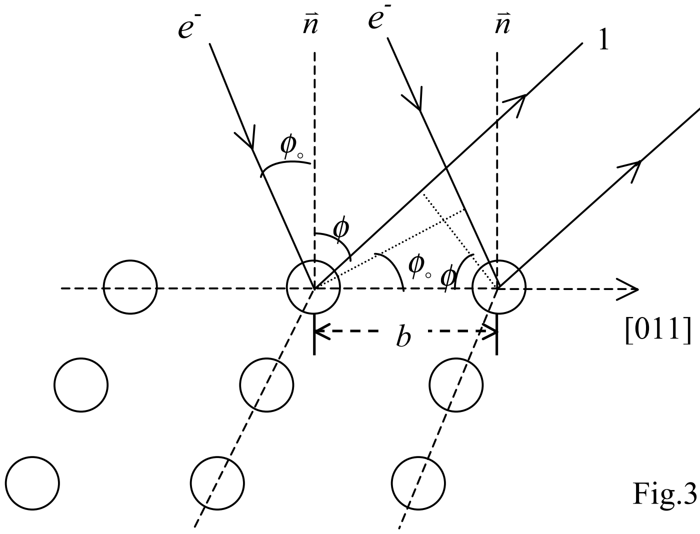


Fig.3c

The path difference between electron beam 1 and 2 is

$$\Delta\ell = b(\sin\phi - \sin\phi_\circ) = n\lambda$$

Given $\phi_\circ = 15.0^\circ$, $\lambda =$ $1.53\,\mathring{A}$ and $b = \frac{a}{\sqrt{2}} = \frac{3.92}{\sqrt{2}} = 2.77\,\mathring{A}$, two solutions are possible.

(i) When $n = 0$, $\phi = \phi_\circ = 15.0^\circ$ (Answer 1)

(ii) When $n = 1$

$$\Delta\ell = 2.77(\sin\phi - \sin 15^\circ) = 1\times1.53$$
$$\sin\phi = \frac{1.53 + 0.72}{2.77} = 0.812$$

*[**Solution**] (continued)* *Theoretical Question 3*

Vibration of Surface Atoms

$$\phi = 54.3^\circ \qquad \text{(Answer 2)}$$

For $n = 2$, no solution exists as $\Delta\ell = 2.77(sin\phi - sin 15^\circ) = 2\times1.53$ and $sin\phi > 1$.

(2) $I = I_0\, exp\langle -(\vec{u}\cdot\Delta\vec{K})^2\rangle$

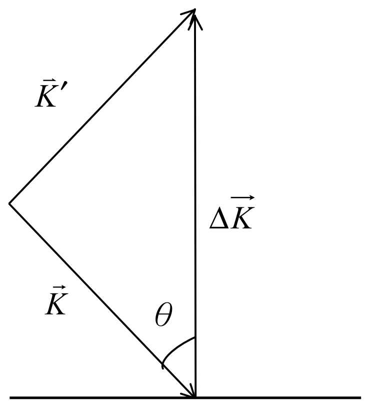


Fig. 3d

For the specularly reflected beam, we have from Fig. 3d

$$\Delta\vec{K} = \vec{K}' - \vec{K} = 2K\cos\theta \quad \hat{x}$$

where $\hat{x}$ is the unit vector in the direction of the surface normal. Take the x-component of $\vec{u}$, we then obtain

$$I = I_0 e^{-<u_x^2(t)\cdot 4K^2 cos^2\theta>} = I_0 e^{-4K^2 cos^2\theta <u_x{}^2(t)>} \qquad (2)$$

The vibration in the direction of the surface normal of the surface atoms is simple harmonic, take

$$u_x(t) = A\cos\omega t$$

$$\because \qquad \langle u_x{}^2(t)\rangle = \frac{1}{\tau}\int_0^\tau u^2 dt = \frac{1}{\tau}\int_0^\tau A^2 cos^2\,\omega t\, dt = \frac{A^2}{\tau}\cdot\frac{\tau}{2} = \frac{A^2}{2}$$

$$\therefore \quad A^2 = 2\langle u_x{}^2(t)\rangle$$

The total energy E is thus given by

$$E = \frac{1}{2}CA^2 = \frac{1}{2}C\cdot 2 < u_x{}^2(t) > \; = \; C < u_x{}^2(t) > \; = \; m'\omega^2 < u_x{}^2(t) >$$

*[**Solution**] (continued)* *Theoretical Question 3*

Vibration of Surface Atoms

Therefore, one obtains

$$< u_x^{\,2}(t) > \; = \; E \,/\, (\, m'\omega^2)$$

$$E = m'\omega^2 < u_x^{\,2} > = k_B T$$

where m' is the mass of the atom. From either of the above two equations, one then has the following equality

$$< u_x^{\,2} > = \frac{k_B T}{m'\omega^2} = \frac{k_B T}{m' 4\pi^2 f^2} \qquad (3)$$

From eq. (3) and eq. (2), one obtains

$$I = I_0 e^{-4K^2 \cos^2\theta \frac{k_B T}{m' 4\pi^2 f^2}}$$

where $K = \dfrac{2\pi\, p}{h} = \dfrac{2\pi}{\lambda}$. Accordingly,

$$I = I_0 e^{-\frac{4k_B \cos^2\theta}{m' f^2 \lambda^2} T} = I_0 e^{-M'T} \qquad (4)$$

and

$$\ell n \frac{I}{I_0} = -M'T$$

From the plot of $\ell n \dfrac{I}{I_0}$ versus T, one obtains the slope

$$M' = \frac{4k_B \cos^2\theta}{m' f^2 \lambda^2} \qquad (5)$$

The slope of the curve can be estimated from Fig. 3b and leads to the result

$$M' = 2.3 \times 10^{-3}.$$

Using the following data in Eq.(5),

$$k_B = 1.38 \times 10^{-23}\, J/K$$

$$\lambda = 1.53 \times 10^{-10}\, m$$

$$m' = 195.1 \times 10^{-3} / (6.02 \times 10^{23}) = 3.24 \times 10^{-25} \;\; kg/atom$$

[Solution] (continued) *Theoretical Question 3*

Vibration of Surface Atoms

one finds

$$2.3\times10^{-3} = \frac{4\times1.38\times10^{-23}\cdot cos^2 15^\circ}{\frac{195.1\times10^{-3}}{6.02\times10^{23}}\times f^2\times(1.53\times10^{-10})^2}$$

The solution for frequency is then

$$f^2 = 3.0\times10^{24}\,(\text{new}) \Rightarrow \quad f = 1.7\times10^{12}\,Hz \qquad \text{Answer (a)}$$

From $< u_x{}^2 >= \dfrac{k_B T}{m' 4\pi^2 f^2}$, $T = 300K$, one finally obtains

$$< u_x{}^2 >= \frac{1.38\times10^{-23}\cdot 300}{\frac{195.1\times10^{-3}}{6.02\times10^{23}}\times 4\pi^2\times 3.0\times10^{24}} = 1.1\times10^{-22}\,m^2\,(\text{new})$$

and

$$\sqrt{< u_x{}^2 >} = 1.0\times10^{-11}\,m = 0.10\,\mathring{A}\,(\text{new})$$

Ans