

Solution 1

(a) $m\ddot{X}_n = S(X_{n+1} - X_n) - S(X_n - X_{n-1})$.	0.7
(b) Let $X_n = A \sin nka \cos(\omega t + \alpha)$, which has a harmonic time dependence. By analogy with the spring, the acceleration is $\ddot{X}_n = -\omega^2 X_n$.	
Substitute into (a): $-mA\omega^2 \sin nka = AS \{\sin(n+1)ka - 2 \sin nka + \sin(n-1)ka\}$	
$= -4SA \sin nka \sin^2 \frac{1}{2}ka$.	0.6
Hence $\omega^2 = (4S/m) \sin^2 \frac{1}{2}ka$.	0.2
To determine the allowed values of k , use the boundary condition $\sin(N+1)ka = \sin kL = 0$.	0.7
The allowed wave numbers are given by $kL = \pi, 2\pi, 3\pi, \dots, N\pi$ (N in all),	0.3
and their corresponding frequencies can be computed from $\omega = \omega_0 \sin \frac{1}{2}ka$,	
in which $\omega_{\max} = \omega_0 = 2(S/m)^{1/2}$ is the maximum allowed frequency.	0.4
(c) $\langle E(\omega) \rangle = \frac{\sum_{p=0}^{\infty} p \hbar \omega P_p(\omega)}{\sum_{p=0}^{\infty} P_p(\omega)}$	
First method: $\frac{\sum_{n=0}^{\infty} n \hbar \omega e^{-n \hbar \omega / k_B T}}{\sum_{n=0}^{\infty} e^{-n \hbar \omega / k_B T}} = k_B T^2 \frac{\partial}{\partial T} \ln \sum_{n=0}^{\infty} e^{-n \hbar \omega / k_B T}$	1.5
The sum is a geometric series and is $\{1 - e^{-\hbar \omega / k_B T}\}^{-1}$	0.5
We find $\langle E(\omega) \rangle = \frac{\hbar \omega}{e^{\hbar \omega / k_B T} - 1}$.	
<i>Alternatively:</i> denominator is a geometric series $= \{1 - e^{-\hbar \omega / k_B T}\}^{-1}$	(0.5)
Numerator is $k_B T^2 (d/dT)$ (denominator) $= e^{-\hbar \omega / k_B T} \{1 - e^{-\hbar \omega / k_B T}\}^{-2}$ and result follows.	(1.5)

<i>A non-calculus method:</i> Let $D = 1 + e^{-x} + e^{-2x} + e^{-3x} + \dots$, where $x = \hbar\omega/k_B T$. This is a geometric series and equals $D = 1/(1 - e^{-x})$. Let $N = e^{-x} + 2e^{-2x} + 3e^{-3x} + \dots$. The result we want is N/D. Observe $\begin{aligned} D - 1 &= e^{-x} + e^{-2x} + e^{-3x} + e^{-4x} + e^{-5x} + \dots \\ (D - 1)e^{-x} &= e^{-2x} + e^{-3x} + e^{-4x} + e^{-5x} + \dots \\ (D - 1)e^{-2x} &= e^{-3x} + e^{-4x} + e^{-5x} + \dots \end{aligned}$ Hence $N = (D - 1)D$ or $N/D = D - 1 = \frac{e^{-x}}{1 - e^{-x}} = \frac{1}{e^x - 1}$.	(2.0)
(d) From part (b), the allowed k values are $\pi/L, 2\pi/L, \dots, N\pi/L$.	
Hence the spacing between allowed k values is π/L , so there are $(L/\pi)\Delta k$ allowed modes in the wave-number interval Δk (assuming $\Delta k \gg \pi/L$).	1.0
(e) Since the allowed k are $\pi/L, \dots, N\pi/L$, there are N modes.	0.5
Follow the problem: $d\omega/dk = a\omega_0 \cos ka$ from part (a) & (b) $= \frac{1}{2}a\sqrt{\omega_{\max}^2 - \omega^2}$, $\omega_{\max} = \omega_0$. This second form is more convenient for integration. The number of modes dn in the interval $d\omega$ is	0.5
$\begin{aligned} dn &= (L/\pi)\Delta k = (L/\pi) (dk/d\omega) d\omega \\ &= (L/\pi) \{a\omega_0 \cos ka\}^{-1} d\omega \\ &= \frac{L}{\pi} \frac{2}{a} \frac{1}{\sqrt{\omega_{\max}^2 - \omega^2}} d\omega \\ &= \frac{2(N+1)}{\pi} \frac{1}{\sqrt{\omega_{\max}^2 - \omega^2}} d\omega \end{aligned}$ Total number of modes $= \int dn = \int_0^{\omega_{\max}} \frac{2(N+1)}{\pi} \frac{d\omega}{\sqrt{\omega_{\max}^2 - \omega^2}} = N + 1 \approx N$ for large N.	0.5 for eitl This part is necessary f E_T below, but not for number of modes (0.5)
Total crystal energy from (c) and dn of part (e) is given by $E_T = \frac{2N}{\pi} \int_0^{\omega_{\max}} \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1} \frac{d\omega}{\sqrt{\omega_{\max}^2 - \omega^2}}.$	0.7
(f) Observe first from the last formula that E_T increases monotonically with temperature since	

$\{e^{\hbar\omega/kT} - 1\}^{-1}$ is increasing with T .

0.2

When $T \rightarrow 0$, the term -1 in the last result may be neglected in the denominator so

0.2

$$E_T \approx_{T \rightarrow 0} \frac{2N}{\pi} \int \hbar\omega e^{-\hbar\omega/k_B T} \frac{1}{\sqrt{\omega_{\max}^2 - \omega^2}} d\omega$$

$$= \frac{2N}{\hbar\pi\omega_{\max}} (k_B T)^2 \int_0^{\infty} \frac{x e^{-x}}{\sqrt{1 - (k_B T x / \hbar\omega_{\max})^2}} dx$$

0.3

0.2

which is quadratic in T (denominator in integral is effectively unity) hence C_V is linear in T near absolute zero.

0.2

Alternatively, if the summation is retained, we have

$$E_T = \frac{2N}{\pi} \sum_{\omega} \frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1} \frac{\Delta\omega}{\sqrt{\omega_{\max}^2 - \omega^2}} \rightarrow_{T \rightarrow 0} \frac{2N}{\pi} \sum_{\omega} \hbar\omega e^{-\hbar\omega/k_B T} \frac{\Delta\omega}{\sqrt{\omega_{\max}^2 - \omega^2}}$$

$$= \frac{2N}{\pi} \frac{(k_B T)^2}{\hbar\omega} \sum_y e^{-y} y \Delta y$$

(0.5)

When $T \rightarrow \infty$, use $e^x \approx 1 + x$ in the denominator,

0.2

$$E_T \approx_{T \rightarrow \infty} \frac{2N}{\pi} \int_0^{\omega_{\max}} \frac{\hbar\omega}{\hbar\omega/k_B T} \frac{1}{\sqrt{\omega_{\max}^2 - \omega^2}} d\omega = \frac{2N}{\pi} k_B T \frac{\pi}{2},$$

0.1

which is linear; hence $C_V \rightarrow Nk_B = R$, the universal gas constant. This is the Dulong-Petit rule.

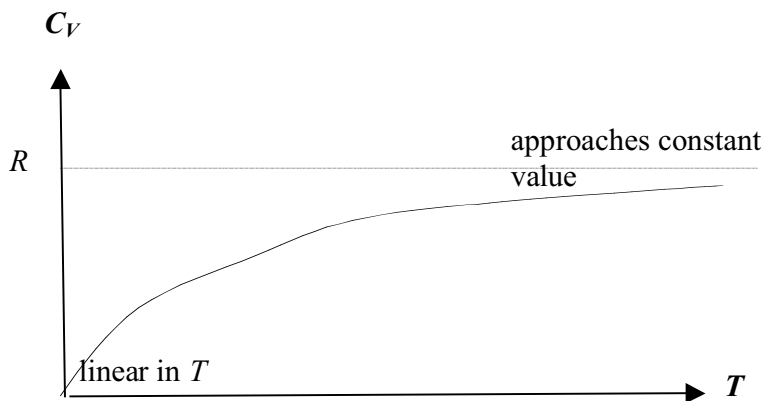
Alternatively, if the summation is retained, write denominator as $e^{\hbar\omega/k_B T} - 1 \approx \hbar\omega/k_B T$ and

(0.2)

$$E_T \rightarrow_{T \rightarrow \infty} \frac{2N}{\pi} k_B T \sum_{\omega} \frac{\Delta\omega}{\sqrt{\omega_{\max}^2 - \omega^2}} \text{ which is linear in } T, \text{ so } C_V \text{ is constant.}$$

Sketch of C_V versus T :

0.5



Answer sheet: Question 1

(a) Equation of motion of the n^{th} mass is:

$$m\ddot{X}_n = S(X_{n+1} - X_n) - S(X_n - X_{n-1}).$$

(b) Angular frequencies ω of the chain's vibration modes are given by the equation:

$$\omega^2 = (4S/m) \sin^2 ka.$$

Maximum value of ω is: $\omega_{\text{max}} = \omega_0 = 2(S/m)^{1/2}$

The allowed values of the wave number k are given by:

$$\pi/L, 2\pi/L, \dots, N\pi/L.$$

How many such values of k are there? N

(f) The average energy per frequency mode ω of the crystal is given by:

$$\langle E(\omega) \rangle = \frac{\hbar \omega}{e^{\hbar \omega / k_B T} - 1}$$

(g) There are how many allowed modes in a wave number interval Δk ?

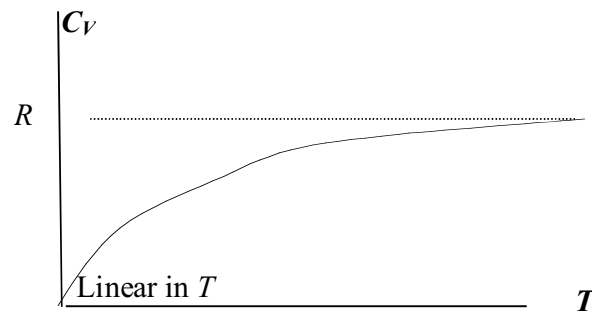
$$(L/\pi)\Delta k.$$

(e) The total number of modes in the lattice is: N

Total energy E_T of crystal is given by the formula:

$$E_T = \frac{2N}{\pi} \int_0^{\omega_{\max}} \frac{\hbar \omega}{e^{\hbar \omega / k_B T} - 1} \frac{d\omega}{\sqrt{\omega_{\max}^2 - \omega^2}}.$$

(h) A sketch (graph) of C_V versus absolute temperature T is shown below.



For $T \ll 1$, C_V displays the following behaviour: C_V is linear in T .

As $T \rightarrow \infty$, C_V displays the following behaviour: $C_V \rightarrow Nk_B = R$, the universal gas constant.

Solution to Question 2: The Rail Gun

<u>Proper Solution (taking induced emf into consideration):</u> (a) Let I be the current supplied by the battery in the absence of back emf. Let i be the induced current by back emf ε_b. Since $\varepsilon_b = d\phi / dt = d(BLx)/dt = BLv$, $\therefore i = BLv / R$. Net current, $I_N = I - i = I - BLv / R$. Forces parallel to rail are: Force on rod due to current is $F_c = BLI_N = BL(I - BLv / R) = BLI - B^2 L^2 v / R$. Net force on rod and young man combined is $F_N = F_c - mg \sin \theta$. (1) Newton's law: $F_N = ma = m dv / dt$. (2) Equating (1) and (2), & substituting for F_c & dividing by m, we obtain the acceleration $dv / dt = \alpha - v / \tau$, where $\alpha = BIL / m - g \sin \theta$ and $\tau = mR / B^2 L^2$.	1 0.5 0.5 0.5 0.5	3
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(b)(i) Since initial velocity of rod = 0, and let velocity of rod at time t be $v(t)$, we have $v(t) = v_{\infty} \left(1 - e^{-t/\tau}\right), \quad (3)$ where $v_{\infty}(\theta) = \alpha\tau = \frac{IR}{BL} \left(1 - \frac{mg}{BLI} \sin \theta\right)$. Let t_s be the total time he spent moving along the rail, and v_s be his velocity when he leaves the rail, i.e. $v_s = v(t_s) = v_{\infty} \left(1 - e^{-t_s/\tau}\right). \quad (4)$ $\therefore t_s = -\tau \ln(1 - v_s / v_{\infty}) \quad (5)$	0.5 0.5 0.5	1.5
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(b) (ii) Let t_f be the time in flight: $t_f = \frac{2v_s \sin \theta}{g} \quad (6)$	0.5	
He must travel a horizontal distance w during t_f. $w = (v_s \cos \theta) t_f \quad (7)$ $t_f = \frac{w}{v_s \cos \theta} = \frac{2v_s \sin \theta}{g} \quad (8) \text{ (from (6) \& (7))}$	0.5	
From (8), v_s is fixed by the angle θ and the width of the strait w $v_s = \sqrt{\frac{gw}{\sin 2\theta}}. \quad (9)$ $\therefore t_s = -\tau \ln \left(1 - \frac{1}{v_\infty} \sqrt{\frac{gw}{\sin 2\theta}} \right), \quad (\text{Substitute (9) in (5)})$		1.5
And $t_f = \frac{2 \sin \theta}{g} \sqrt{\frac{gw}{\sin 2\theta}} = \sqrt{\frac{2w \tan \theta}{g}} \quad (\text{Substitute (9) in (8)})$	0.5	

(c)

Therefore, total time is:
$$T = t_s + t_f = -\tau \ln \left(1 - \frac{1}{v_\infty} \sqrt{\frac{gw}{\sin 2\theta}} \right) + \sqrt{\frac{2w \tan \theta}{g}}$$

The values of the parameters are: $B=10.0$ T, $I= 2424$ A, $L=2.00$ m, $R=1.0 \Omega$, $g=10$ m/s², $m=80$ kg, and $w=1000$ m.

Then
$$\tau = \frac{mR}{B^2 L^2} = \frac{(80)(1.0)}{(10.0)^2 (2.00)^2} = 0.20 \text{ s.}$$

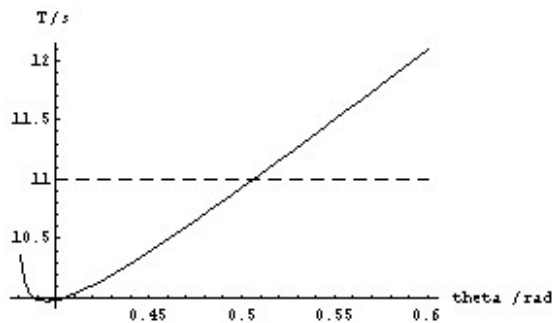
$$v_\infty(\theta) = \frac{2424}{(10.0)(2.00)} \left(1 - \frac{(80)(10)}{(10.0)(2.00)(2424)} \sin \theta \right)$$

$$= 121(1 - 0.0165 \sin \theta)$$

So,

$$T = t_s + t_f = -0.20 \ln \left(1 - \frac{100}{v_\infty} \frac{1}{\sqrt{\sin 2\theta}} \right) + 14.14 \sqrt{\tan \theta}$$

By plotting T as a function of θ , we obtain the following graph:



Note that the lower bound for the range of θ to plot may be determined by the condition $v_s / v_\infty < 1$ (or the argument of $\ln$ is positive), and since mg/BLI is small (0.0165), $v_\infty \approx IR/BL$ ($= 121$ m/s), we have the condition $\sin(2\theta) > 0.68$, i.e. $\theta > 0.37$. So one may start plotting from $\theta = 0.38$.

From the graph, for θ within the range ($\sim 0.38, 0.505$) radian the time T is within 11 s.

Labeling:
0.1 each axis

Unit:
0.1 each axis

Proper Range in θ :
0.3 lower limit
(more than 0.37,
less than 0.5),
0.2 upper limit
(more than 0.5
and less than 0.6)

Proper shape of
curve: 0.2

Accurate
intersection at
 $\theta = 0.5: 0.4$

1.5

(d)

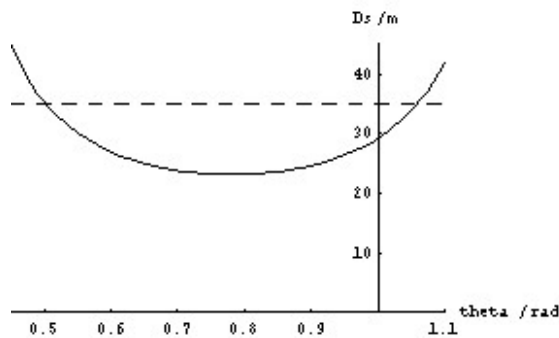
However, there is another constraint, i.e. the length of rail D . Let D_s be the distance travelled during the time interval t_s

$$D_s = \int_0^{t_s} v(t) dt = v_\infty \int_0^{t_s} (1 - e^{-t/\tau}) dt = v_\infty \left(t + \tau e^{-\beta t} \right)_0^{t_s} = v_\infty \left[t_s - \tau (1 - e^{-\beta t_s}) \right] = v_\infty t_s - v(t_s) \tau$$

i.e.

$$D_s = -\tau \left[v_\infty(\theta) \ln \left(1 - \frac{1}{v_\infty(\theta)} \sqrt{\frac{gw}{\sin 2\theta}} \right) + \sqrt{\frac{gw}{\sin 2\theta}} \right]$$

The graph below shows D_s as a function of θ .



It is necessary that $D_s \leq D$, which means θ must range between .5 and 1.06 radians.

In order to satisfy both conditions, θ must range between 0.5 & 0.505 radians.

(Remarks: Using the formula for t_f , t_s & D , we get

At $\theta = 0.507$, $t_f = 10.540$, $t_s = 0.466$, giving $T = 11.01$ s, & $D = 34.3$ m

At $\theta = 0.506$, $t_f = 10.527$, $t_s = 0.467$, giving $T = 10.99$ s, & $D = 34.4$ m

At $\theta = 0.502$, $t_f = 10.478$, $t_s = 0.472$, giving $T = 10.95$ s, & $D = 34.96$ m

At $\theta = 0.50$, $t_f = 10.453$, $t_s = 0.474$, giving $T = 10.93$ s, & $D = 35.2$ m,

So the more precise angle range is between 0.502 to 0.507, but students are not expected to give such answers.

To 2 sig fig $T = 11$ s. Range is 0.50 to 0.51 (in degree: 28.6° to 29.2° or 29°)

0.5

Labeling:
0.1 each axis

Unit:
0.1 each axis

Proper Range in θ :
0.3 lower limit
(more than 0.4,
less than 0.49),
0.2 upper limit
(more than 0.51
and less than 1.1)

Proper shape of
curve: 0.2

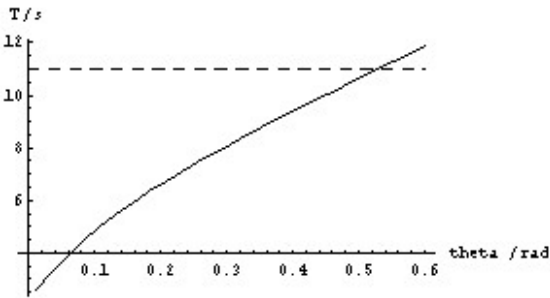
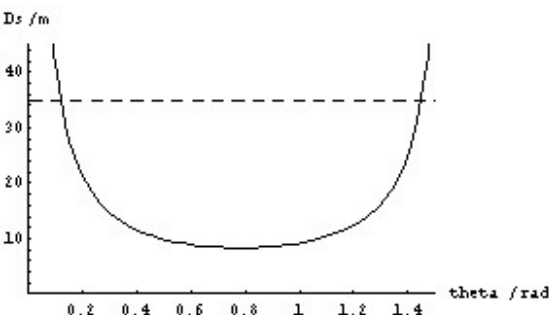
Accurate
intersection at
 $\theta = 0.5$: 0.4

0.5

2.5

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<u>Alternate Solution (Not taking induced emf into consideration):</u> If induced emf is not taken into account, there is no induced current, so the net force acting on the combined mass of the young man and rod is $F_N = BIL - mg \sin \theta .$ And we have instead $dv/dt = \alpha,$ where $\alpha = BIL/m - g \sin \theta .$ $\therefore v(t) = \alpha t$ and $\therefore v_s = v(t_s) = \alpha t_s$ $t_f = \frac{2v_s \sin \theta}{g} = \frac{2\alpha t_s \sin \theta}{g} .$ Therefore, $w = (v_s \cos \theta) t_f = \frac{\alpha^2 t_s^2 \sin 2\theta}{g} ,$ giving $t_s = \frac{1}{\alpha} \sqrt{\frac{gw}{\sin 2\theta}}$ and $t_f = \sqrt{\frac{2w \tan \theta}{g}} .$ Hence, $T = t_s + t_f = \frac{1}{\alpha} \sqrt{\frac{gw}{\sin 2\theta}} + \sqrt{\frac{2w \tan \theta}{g}} = \frac{\sqrt{wg}}{\alpha} \left[1 + 2 \left(\frac{\alpha}{g} \right) \sin \theta \right] .$ where $\alpha = BIL/m - g \sin \theta .$ The values of the parameters are: B=10.0 T, I= 2424 A, L=2.00m, R=1.0 Ω, g=10 m/s², m=80 kg, and w=1000 m. Then, $T = \frac{100}{\alpha} \frac{[1 + 0.20\alpha \sin \theta]}{\sqrt{\sin 2\theta}}$ where $\alpha = 606 - 10 \sin \theta .$	0.2 BIL 0.2 mg sin θ 0.1 0.2 0.5 0.5 0.3	2
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 For θ within the range ($\sim 0, 0.52$) radian the time T is within 11 s.	Labeling: 0.1 each axis Unit: 0.1 each axis Proper Range in θ: 0.1 lower limit (more than 0, less than 0.5), 0.2 upper limit (more than 0.52 and less than 0.8) Proper shape of curve: 0.2 Accurate intersection at $\theta = 0.52$: 0.4	1.3
However, there is another constraint, i.e. the length of rail D. Let D_s be the distance travelled during the time interval t_s $D_s = \frac{gw}{2\alpha \sin 2\theta} = \frac{5000}{\alpha \sin 2\theta}$ which is plotted below  It is necessary that $D_s \leq D$, which means θ must range between 0.11 and 1.43 radians. In order to satisfy both conditions, θ must range between 0.11 & 0.52 radians.	Labeling: 0.1 each axis Unit: 0.1 each axis Proper Range in θ: 0.1 lower limit (more than 0.08, less than 0.11), 0.1 upper limit (more than 0.52 and less than 1.5) Proper shape of curve: 0.2 Accurate intersection at $\theta = 0.11$: 0.4	1.2 0.5

Question 3 - Marking Scheme

(a) Since $W(v) = 4\pi \left(\frac{M}{2\pi R T} \right)^{3/2} v^2 e^{-M v^2 / (2RT)}$,

$$\begin{aligned} \bar{v} &= \int_0^{\infty} v W(v) dv = \\ &= \int_0^{\infty} v 4\pi \left(\frac{M}{2\pi R T} \right)^{3/2} v^2 e^{-M v^2 / (2RT)} dv \\ &= \int_0^{\infty} 4\pi \left(\frac{M}{2\pi R T} \right)^{3/2} v^3 e^{-M v^2 / (2RT)} dv \\ &= 4\pi \left(\frac{M}{2\pi R T} \right)^{3/2} \int_0^{\infty} v^3 e^{-M v^2 / (2RT)} dv \\ &= 4\pi \left(\frac{M}{2\pi R T} \right)^{3/2} \frac{4 R^2 T^2}{2 M^2} \\ &= \sqrt{\frac{8 R T}{\pi M}} \end{aligned}$$

Marking Scheme:

Performing the integration correctly:

1 mark

Simplifying

0.5 marks

Subtotal for the section

1.5

marks

- (b) Assuming an ideal gas, $P V = N k T$, so that the concentration of the gas molecules, n , is given by

$$n = \frac{N}{V} = \frac{P}{k T}$$

the impingement rate is given by

$$\begin{aligned} J &= \frac{1}{4} n \bar{v} \\ &= \frac{1}{4} \frac{P}{k T} \sqrt{\frac{8 R T}{\pi M}} \\ &= P \sqrt{\frac{8 R T}{16 k^2 T^2 \pi M}} \\ &= P \sqrt{\frac{N_A k}{2 k^2 T \pi M}} \\ &= P \sqrt{\frac{1}{2 k T \pi m}} \\ &= \frac{P}{\sqrt{2 \pi m k T}} \end{aligned}$$

where we have note that $R = N_A k$ and $m = \frac{M}{N_A}$ (N_A being Avogadro number).

Marking Scheme:

Using ideal gas formula to estimate concentration of gas molecules: marks	0.7
Simplifying expression: marks	0.4
Using $R = N k$, and the formula for m ; (0.2 mark each) marks	0.4
<u>Subtotal for the section</u>	<u>1.5</u>
<u>marks</u>	

- (c) Assuming close packing, there are approximately 4 molecules in an area of 16 r^2 m^2 . Thus, the number of molecules in 1 m^2 is given by

$$n_1 = \frac{4}{16 (3.6 \times 10^{-10})^2} = 1.9 \times 10^{18} \text{ m}^{-2}$$

However at $(273 + 300) \text{ K}$ and 133 Pa , the impingement rate for oxygen is

$$\begin{aligned} J &= \frac{P}{\sqrt{2 \pi m k T}} \\ &= \frac{133}{\sqrt{2 \pi \left(\frac{32 \times 10^{-3}}{6.02 \times 10^{23}} \right) (1.38 \times 10^{-23}) 573}} \\ &= 2.6 \times 10^{24} \text{ m}^{-2} \text{ s}^{-1} \end{aligned}$$

Therefore, the time needed for the deposition is $\frac{n_1}{J} = 0.7 \mu\text{s}$

The calculated time is too short compared with the actual processing.

Marking Scheme:

Estimation of number of molecules in 1 m^2 :	0.4 marks
Calculation the impingement rate:	0.6 marks
Taking note of temperature in Kelvin	0.3 marks
Calculating the time	0.4 marks
<u>Subtotal for the section</u>	<u>1.7</u>
<u>marks</u>	

- (d) With activation energy of 1 eV and letting the velocity of the oxygen molecule at this energy is v_1 , we have

$$\frac{1}{2} m v_1^2 = 1.6 \times 10^{-19} \text{ J}$$

$$\Rightarrow v_1 = 2453.57 \text{ ms}^{-1}$$

At a temperature of 573 K, the distribution of the gas molecules is

We can estimate the fraction of the molecules with speed greater than 2454 ms^{-1} using the trapezium rule (or any numerical techniques) with ordinates at 2453, $2453 + 500$, $2453 + 1000$. The values are as follows:

Velocity, v	Probability, $W(v)$
2453	1.373×10^{-10}
2953	2.256×10^{-14}
3453	6.518×10^{-19}

Using trapezium rule, the fraction of molecules with speed greater than 2453 ms^{-1} is given by

$$\text{fraction of molecules} = \frac{500}{2} \left[(1.373 \times 10^{-10}) + (2 \times 2.256 \times 10^{-14}) + (6.518 \times 10^{-19}) \right]$$

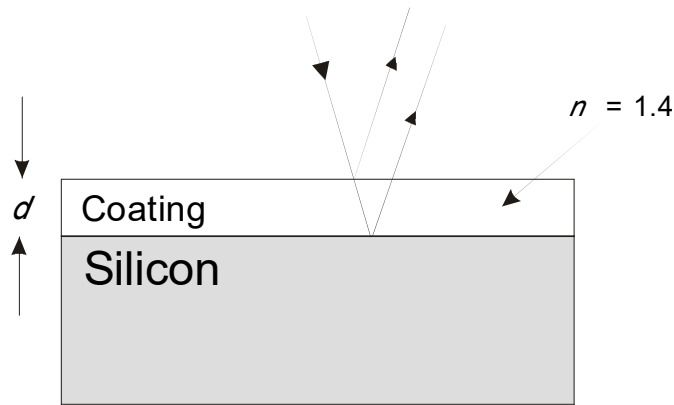
$$f = 3.43 \times 10^{-8}$$

Thus the time needed for the deposition is given by $0.7 \mu\text{s} / (3.43 \times 10^{-8})$ that is 20.4 s

Marking Scheme

Computing the value of the cut-off energy or velocity:	0.6
marks	
Estimating the fraction of molecules	1.2 marks
Correct method of final time	0.4 marks
Correct value of final time	0.6 marks
<u>Subtotal for the section</u>	<u>2.8</u>
<u>marks</u>	

- (e) For destructive interference, optical path difference $= 2 d = \frac{\lambda'}{2}$ where $\lambda' = \frac{\lambda_{\text{air}}}{n}$ is the wavelength in the coating.



The relation is given by:

$$d = \frac{\lambda_{\text{air}}}{4 n}$$

Plugging in the given values, one gets $d = 105$ or 105.2 nm.

Derive equation:

Finding the optical path length 0.2 marks

Knowing that there is a phase change at the reflection 0.5 marks

Putting everything together to get the final expression 0.6 marks

Subtotal: 1.3 marks

Computation of d : 0.6 marks

Getting the correct number of significant figures: 0.6 marks

Subtotal: 1.2 marks

Subtotal for Section 2.5 marks

TOTAL 10 marks