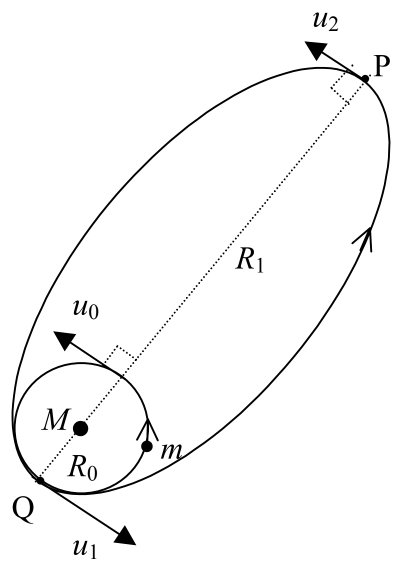
Theoretical Competition

I. Satellite's orbit transfer

In the near future we ourselves may take part in launching of a satellite which, in point of view of physics, requires only the use of simple mechanics.



a) A satellite of mass m is presently circling the Earth of mass M in a circular orbit of radius R_0. What is the speed (u_0) of mass m in terms of M, R_0 and the universal gravitation constant G ? (1 point)

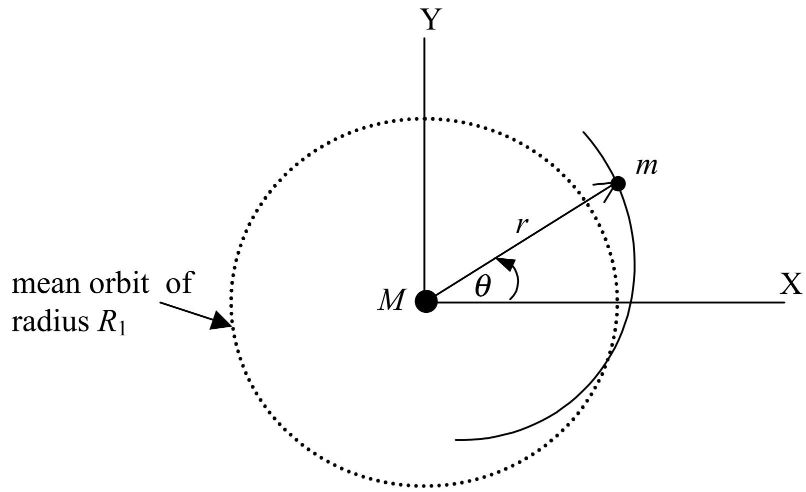
b) We are to put this satellite into a trajectory that will take it to point P at distance R_1 from the centre of the Earth by increasing (almost instantaneously) its velocity at point Q from u_0 to u_1. What is the value of u_1 in terms of u_0, R_0, R_1? (2 points)

c) Deduce the minimum value of u_1 in term of u_0 that will allow the satellite to leave the Earth's influence completely. (1 point)

d) (Referring to part b.) What is the velocity (u_2) of the satellite at point P in terms of u_0, R_0, R_1? (1 point)

e) Now, we want to change the orbit of the satellite at point P into a circular orbit of radius R_1 by raising the value of u_2 (almost instantaneously) to u_3.
What is the magnitude of u_3 in terms of u_2, R_0, R_1? (1 point)

f)



If the satellite is slightly and instantaneously perturbed in the radial direction so that it deviates from its previously perfectly circular orbit of radius R_1, derive the period of its oscillation T of r about the mean distance R_1.

Hint: Students may make use (if necessary) of the equation of motion of a satellite in orbit:

$$m\left[\frac{d^2}{dt^2}r - \left(\frac{d}{dt}\theta\right)^2 r\right] = -G\frac{Mm}{r^2} \qquad \text{.............................. (1)}$$

and the conservation of angular momentum:

$$mr^2\frac{d}{dt}\theta = \text{constant} \qquad \text{.............................. (2)}$$

(3 points)

g) Give a rough sketch of the whole perturbed orbit together with the unperturbed one.

(1 point)

II. Optical Gyroscope

In 1913 Georges Sagnac (1869-1926) considered the use of a ring resonator to search for the aether drift relative to a rotating frame. However, as often happen, his results turned out to be useful ways that Sagnac himself never dreamt of. One of those applications is the Fibre-Optic Gyroscope (FOG) which is based upon a simple phenomena, first observed by Sagnac. The essential physics associated with the Sagnac effect is due to the phase shift caused by two coherent beams of light being sent around a rotating ring of optical fibre in the opposite directions. This phase shift is also used to determine the angular speed of the ring.

As shown in a schematic diagram in Fig. 1, a light wave enters a circular optical fibre light path of radius R at point P on the rotating platform with a uniform angular speed Ω, in the clockwise direction. Here the light wave is split into two waves which travels in the opposite directions, clockwise (CW) and counter clockwise (CCW), through the ring. The refractive index of optical fibre material is μ. Assuming the light traveling inside the fibre-optic cable is a smooth circular path of radius R.

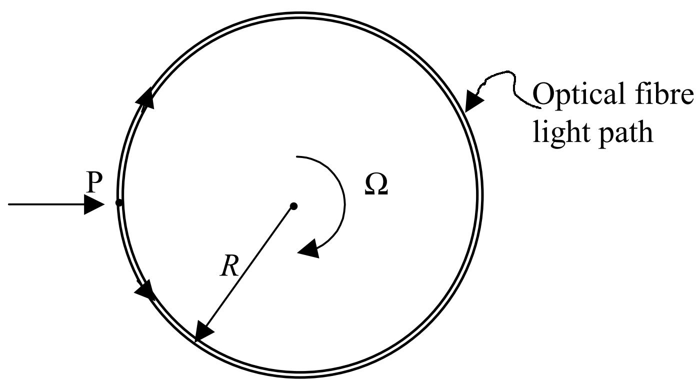


Fig.1

a) Practically, the orbital speed of the ring is much less than the speed of light such that $(R\Omega)^2 << c^2$, find the time difference $\Delta t = t^+ - t^-$ where t^+ and t^- denote the round-trip transit time of the CW and CCW beam respectively. Give your answer in term of area A enclosed by the ring. (2 points)

b) Find the optical path difference, ΔL, for the CW and CCW beams to complete one round-trip of the light within the rotating ring. (2 points)

c) For a circular fibre-optic of radius, $R = 1$ m, what is the maximum value of ΔL for the rotation of the earth? Given $\mu = 1.5$. (1 point)

d) In part b), the measurement could be amplified by increasing number of turns in fibre-optic coil, N, find the phase difference, $\Delta\theta$, for lights to complete the turns. (1 point)

The second scheme of the Optical Gyroscope is Ring Laser Gyroscope (RLG). This could be accomplished by the inclusion of active laser cavity into an equilateral triangular ring, of total length L, as illustrated in Fig.2. The laser source here will generate two amplified coherent light sources propagating in the opposite directions. *In order to sustain the laser oscillation in this triangular ring resonator, the perimeter of the ring must be equal to the integer multiple of wavelength* λ. Etalon, additional component inserted into the ring, is possible to cause frequency selective losses in the ring resonator, so that the undesired modes can be damped and suppressed.

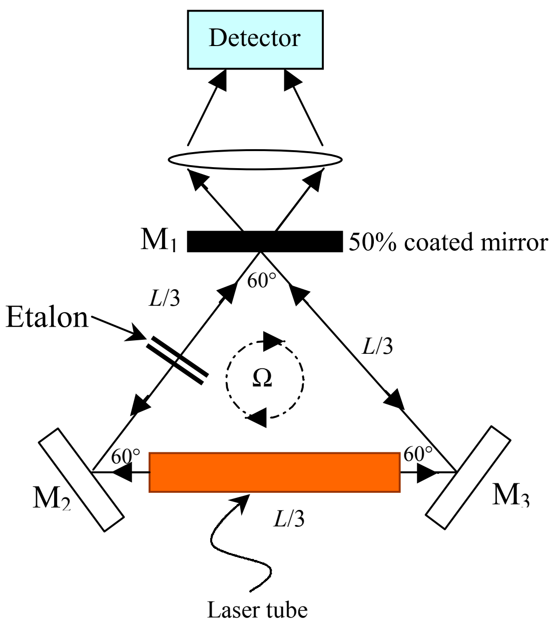


Fig.2: Schematic illustration of the Ring Laser Gyroscope

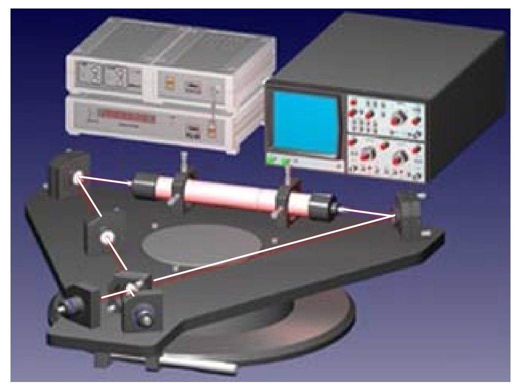


Fig. 3: Illustration of the Ring Laser Gyroscope discussed in this problem

e) Find the time difference of the transit in clockwise and counterclockwise, Δt, for the case of the triangular ring as shown in the figure 2. Give the answer in terms of Ω and the area A enclosed by the ring. Show that this result is the same as that of the circular ring. (2 points)

f) If the ring is rotating with an angular frequency Ω as shown in Fig. 2, there will be frequency difference between CW and CCW measurements. What is the observed beat frequency, $\Delta \nu$, between the CW and CCW beams in terms of L, Ω, λ. (2 points)

III. Plasma Lens

The physics of the intense particle beams has a great impact not only on the basic research but also the applications in medicine and industry. The plasma lens is a device to provide an ultra-strong final focus at the end of the linear colliders. To appreciate the possibilities of the plasma lens it is quite natural to compare this with the usual magnetic and electrostatic lenses. In magnetic lenses, focusing capability is proportional to the magnetic field gradient. The practical upper limit of the quadrupole focusing lens is in the order of 10^2 T/m, while for plasma lens having density of 10^{17} cm^{-3}, its focusing capability is equivalent to a magnetic field gradient of 3×10^6 T/m (about four orders of magnitude more than that of a magnetic quadrupole lens).

In what follows, we will illuminate why the intense relativistic particle beams could produce self-focusing beams and do not blow themselves apart in free space.

a) Consider a long cylindrical electron beam of uniform number density n and average speed v (both quantities in laboratory frame). Derive the expression for the electric field at a point at distance r from the central axis of the beam using classical electromagnetics. (1 point)

b) Derive the expression for the magnetic field at the same point as in a). (2 points)

c) What is then the net outward force on the electron in the electron beam passing that point? (1 point)

d) Assuming that the expression obtained in c) is applicable at relativistic velocities, what will be the force on the electron as v approaches speed of light c, where $c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$? (1 point)

e) If the electron beam of radius R enters into a plasma of uniform density $n_0 < n$ (the plasma is an ionized gas of ions and electrons with equal charge density), what will be the net force on the *stationary plasma ion* at distance r' outside the electron beam long after the beam entering the plasma. You may assume that the density of the plasma ions remains constant and the cylindrical symmetry is maintained. (3 points)

f) After long enough time, what is the net force on an electron at distance r from the central axis of the beam in the plasma, assuming $v \to c$ provided that the density of the plasma ions remains constant and the cylindrical symmetry is maintained? (2 points)