Solution and Marking Scheme

Theory

I. Satellite's orbit transfer

a) $$\frac{mu_0^{\ 2}}{R_0} = \frac{GMm}{R_0^{\ 2}}, \qquad u_0 = \sqrt{\frac{GM}{R_0}}$$ (1 point)

b) conservation of angular momentum: $mu_1R_0 = mu_2R_1$

conservation of energy: $\frac{1}{2}mu_2^{\ 2} - \frac{GMm}{R_1} = \frac{1}{2}mu_1^{\ 2} - \frac{GMm}{R_0}$

$$\left[\left(\frac{R_0}{R_1}\right)^2 - 1\right]u_1^{\ 2} = 2GM\left[\frac{1}{R_1} - \frac{1}{R_0}\right]$$

$$\frac{(R_0 - R_1)(R_0 + R_1)}{R_1^{\ 2}}u_1^{\ 2} = (2GM)\frac{(R_0 - R_1)}{R_0R_1}$$

$$u_1 = \sqrt{\frac{GM}{R_0}}\sqrt{\frac{2R_1}{R_1 + R_0}} = u_0\sqrt{\frac{2R_1}{R_1 + R_0}}$$ (2 points)

c) $$\lim_{R_1 \to \infty} u_1 = \sqrt{2}\,u_0$$ (1 point)

d) $$u_2 = u_1\frac{R_0}{R_1} = u_0\frac{\sqrt{2}\,R_0}{\sqrt{R_1(R_1 + R_0)}}$$ (1 point)

e) $$u_3 = \sqrt{\frac{GM}{R_1}} = \sqrt{\frac{GM}{R_0}}\sqrt{\frac{R_0}{R_1}} = u_0\sqrt{\frac{R_0}{R_1}}$$

$$= \sqrt{\frac{R_0}{R_1}}\sqrt{\frac{R_1(R_1 + R_0)}{\sqrt{2}R_0}}\,u_2$$

$$u_3 = u_2\sqrt{\frac{R_1 + R_0}{2R_0}}$$ (1 point)

f) (3 points) combining equations (1) and (2) :

$$\frac{d^2}{dt^2}r - \frac{C/m}{r^3} = -\frac{GM}{r^2}$$

and for the circular orbit of radius R_1 we have $\frac{C}{m} = GMR_1$

hence $$\frac{d^2}{dt^2}r - \frac{GMR_1}{r^3} = -\frac{GM}{r^2}$$

putting $r = R_1 + \eta$, where $\eta << R_1$

$$\therefore\ \frac{d^2}{dt^2}\eta - \frac{GMR_1}{R_1^{\ 3}\left(1 + \frac{\eta}{R_1}\right)^3} = -\frac{GM}{R_1^{\ 2}\left(1 + \frac{\eta}{R_1}\right)^2}$$

$$\frac{d^2}{dt^2}\eta - \frac{GM}{R_1^{\ 2}}\left(1 - 3\frac{\eta}{R_1}\right) \approx -\frac{GM}{R_1^{\ 2}}\left(1 - 2\frac{\eta}{R_1}\right)$$

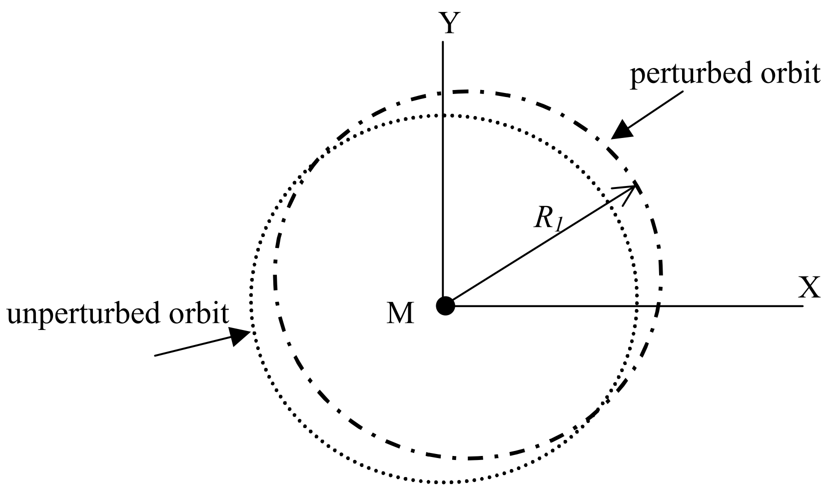
$$\frac{d^2}{dt^2}\eta \approx -\frac{GM}{R_1^{\ 3}}\eta$$

the frequency of oscillation about mean distance is $f = \dfrac{1}{2\pi}\sqrt{\dfrac{GM}{R_1^{\ 3}}}$

the period $T = \dfrac{1}{f} = 2\pi\sqrt{\dfrac{R_1^{\ 3}}{GM}}$

Note that this period is the same as the orbital period

h) (1 point)



Solution and Marking Scheme

Theory

II. Optical Gyroscope

The light wave moves with speed $c' = \frac{c}{\mu}$ in the medium having refractive index μ. Wavelength of light in medium $\lambda' = \frac{\lambda}{\mu}$, where λ is the wavelength of light in vacuum.

a) (2 points)

transit time for the CW beam: $t^+ = \frac{2\pi R + R\Omega t^+}{c'} = \frac{2\pi R}{c'}(1 - \frac{R\Omega}{c'})^{-1}$

transit time for the CCW beam: $t^- = \frac{2\pi R - R\Omega t^-}{c'} = \frac{2\pi R}{c'}(1 + \frac{R\Omega}{c'})^{-1}$

the time difference between t^+ and t^-: $\Delta t = \frac{4\pi R^2 \Omega}{(c')^2 - R^2\Omega^2}$

since $(R\Omega)^2 << (c')^2$ $\quad \Delta t \approx \frac{4\pi R^2 \Omega}{(c')^2}$

b) (2 points) the round-trip optical path difference, ΔL, is given by

$$\Delta L = c'\Delta t = \frac{4\pi R^2 \Omega}{c'}$$

c) (1 point) $\Delta L \cong 4.5 \times 10^{-12}$ m.

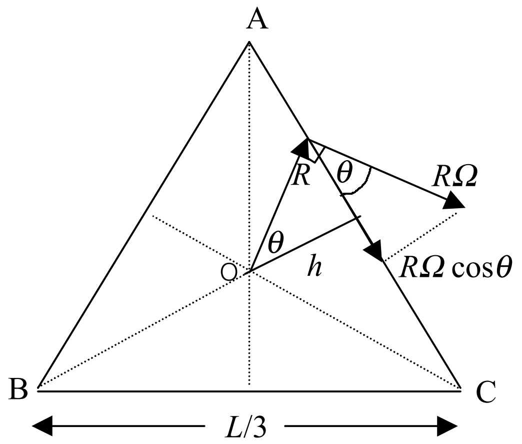
d) (1 point) the corresponding optical phase difference $\Delta\theta$ is,

$$\Delta\theta = \frac{2\pi\Delta L}{\lambda'} = \frac{8\pi^2 R^2 \Omega}{c\lambda'}, \text{ where } \lambda' = \frac{\lambda}{\mu}$$

for N turns of fiber optic ring,

$$\Delta\theta = \frac{8\pi^2 R^2 N\Omega}{c\lambda'}$$

e) (2 points)



The figure shows the triangular ring rotating about the centre o with the angular speed Ω in the clockwise direction. Without loosing generality, let‘s first consider the velocity of light along AC in the CW and CCW direction,

$v_{\pm} = c \pm R\Omega\cos\theta = c \pm \Omega h$, where h is constant.

$$\tau_{\pm} = \frac{L/3}{v_{\pm}} = \frac{L/3}{c \pm \Omega h} \approx \frac{L/3}{c}(1 \mp \frac{\Omega h}{c})$$

where $\tau_{\pm}$ is the time taken for light travelling along AC in the CW and CCW.

$t_{\pm} = \frac{L}{v_{\pm}} = \frac{L}{c \pm \Omega h} \approx \frac{L}{c}(1 \mp \frac{\Omega h}{c})$, where L is the perimeter of the triangular ring.

Therefore, the time difference of light travelling in one complete cycle.

$\Delta t = \frac{2\Omega L h}{c^2} = \frac{4\Omega}{c^2}\left(\frac{1}{2}Lh\right) = \frac{4\Omega A}{c^2}$, where A is the area of the triangular ring.

f) The resonance frequencies associated with $L_{\pm}$ corresponding to the effective cavity lengths seen by CW and CCW propagating beams respectively is,

$$L_{+} = ct^{+} \approx L(1 - \frac{\Omega h}{c})$$

$$L_{-} = ct^{-} \approx L(1 + \frac{\Omega h}{c})$$

where $L_{\pm}$ is the perimeter of the equilateral triangle in the CW (+) and CCW (-) and we also use the fact that $h\Omega << c$. Therefore,

$$\Delta L = L_{-} - L_{+} = 2L\frac{\Omega h}{c} = \frac{4\Omega A}{c} = \frac{\Omega L^2}{\sqrt{3}\, c}$$

The condition to sustain the laser oscillation (given in the problem),

$\nu_{\pm} = \frac{m}{L_{\pm}}c$, m = 1, 2, 3, … integers (1 point)

$\Delta\nu = \nu_{-} - \nu_{+} = \frac{m}{L_{-}}c - \frac{m}{L_{+}}c \approx mc\frac{\Delta L}{L^2} = \nu\frac{\Delta L}{L}$ (1 point) the approximation arises from $L_+L_- \approx L^2$

where L is the perimeter of the triangular ring. Hence,

$$\Delta\nu = \frac{\Delta L}{L}\nu = \frac{4A}{Lc}\nu\Omega = \frac{1}{\sqrt{3}}\frac{L}{\lambda}\Omega \qquad \text{(1 point)}$$

Solution and Marking Scheme

Theory

III. Plasma Lens

a) Consider cylindrical Gaussian surface of radius r and length ℓ about the central axis.
From Gauss's law

$$E 2\pi r\ell = \frac{1}{\varepsilon_0}(\text{charge inside}) \qquad (0.5 \text{ points})$$

$$= -\frac{1}{\varepsilon_0} ne(\pi r^2 \ell)$$

$$E_r = -\frac{ner}{2\varepsilon_0} \qquad (0.5 \text{ point})$$

b) From Ampere's law, $B_\theta = -\dfrac{\mu_0 nerv}{2}$ (2 points)

c) The net Lorentz force is

$$\vec{F} = \left(\frac{ne^2 r}{2\varepsilon_0} - \frac{\mu_0 ne^2 rv^2}{2}\right)\hat{r} = \frac{ne^2 r}{2\varepsilon_0}\left(1 - \frac{v^2}{c^2}\right)\hat{r}$$

where $c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$ (1 point)

d) $F_r \to 0$ as $v \to c$, this implies the electric force and magnetic force cancel each other out. (1 point)

e) The stationary plasma particles have $v = 0$, hence $F_{r'} = \pm eE_{r'}$

where $E_{r'} = -\dfrac{neR^2}{2\varepsilon_o r'} + \dfrac{n_o er'}{2\varepsilon_o}$

for positive ion $F_{r'} = -\dfrac{ne^2R^2}{2\varepsilon_o r'} + \dfrac{n_o e^2 r'}{2\varepsilon_o}$

for electron $F_{r'} = \dfrac{ne^2R^2}{2\varepsilon_o r'} - \dfrac{n_o e^2 r'}{2\varepsilon_o}$

and there is no cancellation from the magnetic force.
As a result the plasma electrons will be blown out, and the ions are pulled in. (2 points)

f) The net force on the electron beam in plasma medium is given by,

$$\vec{F} = \frac{ne^2 r}{2\varepsilon_0}\left(1 - \frac{v^2}{c^2}\right)\hat{r} - \frac{n_0 e^2 r}{2\varepsilon_0}\hat{r} \qquad (2 \text{ points})$$

in the limit $v \to c$, $\vec{F} \approx -\dfrac{n_0 e^2 r}{2\varepsilon_0}\hat{r}$ (1 point)