Solution of Problem No. 1
Measure the mass in the weightless state

1/ Formula for period of oscillation :

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m_0}{k}} \qquad (1)$$

One can deduce :

$$m_0 = \frac{k}{4\pi^2}T_0^2 = 25{,}21 \text{ kg} \qquad (2)$$

2/ When the craft orbits the Earth, oscillating system is the spring with one end attached to the chair of mass m_0 and other end attached to he craft with mass M. This system oscillates like an object with mass

$$m_0' = \frac{m_0 M}{m_0 + M} \qquad (3)$$

attached to an end of the spring, other end of the spring is fixed (m_0' is the reduced mass of the system craft-chair). The period T_0' of the system is also given by (1) and (2). We can deduce :

$$\frac{m_0}{m_0'} = (\frac{T_0}{T_0'})^2 = (\frac{1{,}28195}{1{,}27395})^2 \qquad (4)$$

The mass M of the craft can be calculate from (3) :

$$M = \frac{m_0}{\frac{m_0}{m_0'} - 1} = \frac{25{,}21}{(\frac{T_0}{T_0'})^2 - 1} = \frac{25{,}21}{(\frac{1{,}28195}{1{,}27395})^2 - 1} = 2001 \text{ kg} \qquad (5)$$

Let m be the mass of astronaut and chair, the coressponding reduce mass is m':

$$m' = \frac{mM}{m + M} \qquad (6)$$

the expression of m is then :

$$m = \frac{m'}{1 - \frac{m'}{M}}$$

The reduce mass m' can be calculated from the oscillation period T' by using formula (2) :

$$m' = \frac{605{,}6}{4}.(\frac{2{,}33044}{3{,}1416})^2 = 83{,}31 \text{ kg}$$

the true value of the mass m is :

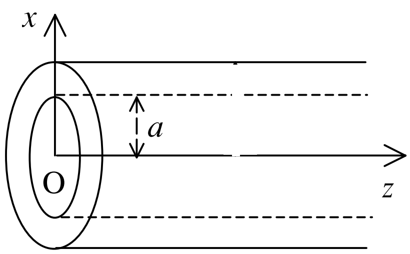
$$m = \frac{83{,}31}{1 - \frac{83{,}31}{2001}} = 86{,}93 kg$$

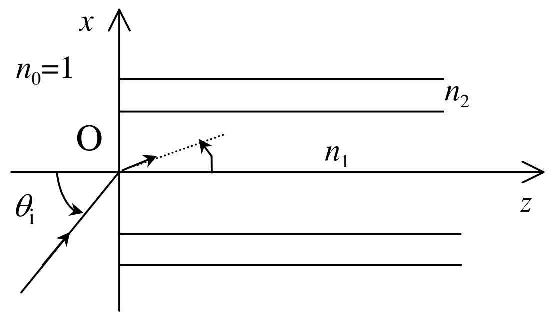
the true value of the mass of astronaut :

$$86{,}93 - 25{,}21 = 61{,}72 \text{ kg}$$

Theoretical Competition-Problem No. 2

Optical fiber





An optical fiber consists of a cylindrical core of radius a, made of a transparent material with refraction index varying gradually from the value $n=n_1$ on the axis to $n = n_2$ (with $1<n_2<n_1$) at a distance a from the axis, according to the formula

$$n = n(x) = n_1\sqrt{1-\alpha^2 x^2}$$

where x is the distance from the core axis and α is a constant. The core is surrounded by a cladding made of a material with constant refraction index n_2. Outside the fiber is air, of refractive index n_0.

Let Oz be the axis of the fiber, with O - the center of the fiber end.

Given n_0=1.000; n_1=1.500; n_2= 1.460, a = 25 μm.

1. A monochromatic light ray enters the fiber at point O under an incident angle θ_i, the incident plane being the plane xOz.

a. Show that at each point on the trajectory of the light in the fiber, the refractive index n and the angle θ between the light ray and the Oz axis satisfy the relationship $n\cos\theta = C$ where C is a constant. Find the expression for C in terms of n_1 and θ_i. ***[1.0 points]***

b. Use the result found in 1.a. and the trigonometric relation $\cos\theta = \left(1+\tan^2\theta\right)^{-\frac{1}{2}}$, where $\tan\theta = \dfrac{dx}{dz} = x'$ is the slope of the tangent to the trajectory at point (x, z), derive an equation for x'. Find the full expression for α in terms of n_1, n_2 and a. By differentiating the two sides of this equation versus z, find the equation for the second derivative x''. ***[1.0 points]***

c. Find the expression of x as a function of z , that is $x = f(z)$, which satisfies the above equation. This is the equation of the trajectory of light in the fiber. ***[1.0 points]***

d. Sketch one full period of the trajectories of the light rays entering the fiber under two different incident angles θ_i. ***[1.0 points]***

2. Light propagates in the optical fiber.

a. Find the maximum incident angle θ_{iM} , under which the light ray still can propagate inside the core of the fiber. ***[1.5 points]***

b. Determine the expression for coordinate z of the crossing points of a light ray with Oz axis for $\theta_i \neq 0$. ***[1.5 points]***

3. The light is used to transmit signals in the form of very short light pulses (of negligible pulse width).

a. Determine the time τ it takes the light to travel from point O to the first crossing point with Oz for incident angle $\theta_i \neq 0$ and $\theta_i \leq \theta_{iM}$.

The ratio of the coordinate z of the first crossing point and τ is called the propagation speed of the light signal along the fiber. Assume that this speed varies monotonously with θ_i.

Find this speed (called v_M) for $\theta_i = \theta_{iM}$.

Find also the propagation speed (called v_0) of the light along the axis Oz.

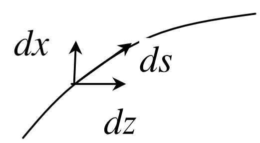
Compare the two speeds. ***[3.25 points]***

b. The light beam bearing the signals is a converging beam entering the fiber at O under different incident angles θ_i with $0 \leq \theta_i \leq \theta_{iM}$. Calculate the highest repetition frequency f of the signal pulses, so that at a distance z = 1000 m two consecutive pulses are still separated (that is, the pulses do not overlap). ***[1.75 points]***

Attention

1. The wave properties of the light are not considered in this problem.
2. Neglect any chromatic dispersion in the fiber.
3. The speed of light in vacuum is c = 2.998×10^8 m/s
4. You may use the following formulae:

- The length of a small arc element ds in the xOz plane is



$$ds = dz\sqrt{1+\left(\frac{dx}{dz}\right)^2}$$

- $$\int \frac{dx}{\sqrt{a^2 - b^2x^2}} = \frac{1}{b} Arc\,\boldsymbol{sin}\frac{bx}{a}$$

- $$\int \frac{x^2dx}{\sqrt{a^2 - b^2x^2}} = -\frac{x\sqrt{a^2 - b^2x^2}}{2b^2} + \frac{a^2 Arc\,\boldsymbol{sin}\frac{bx}{a}}{2b^3}$$

- $\boldsymbol{Arc\,sin}\,x$ is the inverse function of the sine function. Its value equals the less angle the sine of which is x. In other words, if $y = \boldsymbol{Arc\,sin}\,x$ then $\boldsymbol{sin}\,y = x$.

Theoretical Competition-Problem No. 3

Compression and expansion of a two gases system

A cylinder is divided in two compartments with a mobile partition NM. The compartment in the left is limited by the fond of the cylinder and the partition NM (Figure 1). This compartment contains one mole of water vapor. The compartment in the right is limited by the partition NM and a mobile piston AB. This compartment contains one mole of nitrogen gas (N_2).

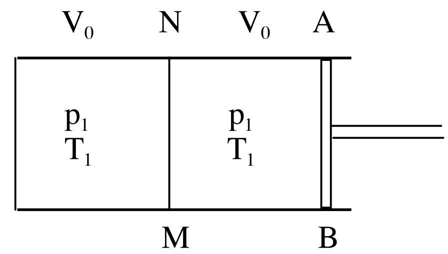


Figure 1

At first, the volumes and temperatures of the gases in two compartments are equal. The partition NM is well heat conductive. His heat capacity is very small and can be neglected.

The specific volume of liquid water is negligible in comparison with the specific volume of water vapor at the same temperature.

The specific latent heat of vaporization L is defined as the amount of heat that must be delivered to one unit of mass of substance to convert it from liquid state to vapor at the same temperature. For water at $T_0 = 373$ K, $L = 2250$ kJ/kg.

1. Suppose that the piston and the wall of the cylinder are heat conductive, the partition NM can slide freely without friction. The initial state of the gases in the cylinder is defined as follows:

Pressure $p_1 = 0.5$ atm.; total volume $V_1 = 2V_0$; temperature $T_1 = 373$ K.

The piston AB slowly compresses the gases in a quasi-static (quasi-equilibrium) and isothermal process to the final volume $V_F = V_0/4$

a. Draw the $p(V)$ curve, that is the curve representing the dependence of pressure p on the total volume V of both gases in the cylinder at temperature T_1. Calculate the coordinates of important points of the curve. ***[1.5 pts]***

Gas constant: $R = 8.31$ J/mol.K or $R = 0.0820$ L.atm./mol.K

1 atm. =101.3 kPa;

Under the pressure p_0 =1 atm., water boils at the temperature $T_0 = 373$ K.

b. Calculate the work done by the piston in the process of gases compressing. ***[1.0 pts]***

$$\int \frac{dV}{V} = \ln V$$

c. Calculate the heat delivered to outside in the process. ***[1.5 pts]***

2. All conditions as in 1. except that there is friction between partition NM and the wall of the cylinder so that NM displaces only when the difference of the pressures acting on its two opposed faces attains 0.5 atm. and over (assuming that the coefficients of static and kinetic friction are equal).

a. Draw the p(V) curve representing the pressure p in the right compartment as a function of the total volume V of both gases in the cylinder at a constant temperature T_1. ***[1.5 pts]***

b. Calculate the work done by the piston in compressing the gases. ***[0.5 pts]***

c. After the volume of gases reaches the value $V_F = V_0/4$, piston AB displaces slowly to the right and makes a quasi-static and isothermal process of expansion of both substances (water and nitrogen) to the initial total volume $2V_0$. Continue to draw in the diagram in question 2.a. the curve representing this process ***[2.0 pts]***

Hint for 2.

Create a table like the one shown here and use it to draw the curves as required in 2.a. and 2.c.

State	Left compartment Volume I Pressure	Right compartment Volume I Pressure	Total volume	Pressure on piston AB
initial	V_0 I 0.5 atm.	V_0 I 0.5 atm.	$2V_0$	0.5 atm.
2	I	I		
3	I	I		
.	I	I		
.	I	I		
.	I	I		
.	I	I		
.	I	I		
final	I	I	$2V_0$	

3. Suppose that the wall and the fond of the cylinder and the piston are heat insulator, the partition NM is fixed and heat conductive, the initial state of gases is as in 1. Piston AB moves slowly toward the right side and increases the volume of the right compartment until the water vapor begins to condense in the left compartment.
   a. Calculate the final volume of the right compartment. ***[3 pts]***
   b. Calculate the work done by the gas in this expansion. ***[1 pts]***

The ratio of isobaric heat capacity to isochoric one $\gamma = \dfrac{C_p}{C_V}$ for nitrogen is $\gamma_1 = \dfrac{7}{5}$ and for water vapor $\gamma_2 = \dfrac{8}{6}$.

In the interval of temperature from 353 K to 393 K one can use the following approximate formula:

$$p = p_0 \boldsymbol{exp}\left[-\frac{\mu L}{R}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right]$$

where T is boiling temperature of water under pressure p, μ - its molar mass. p_0, L and T_0 are given above.