Theoretical Question 1

Laser cooling of atoms

In this problem you are asked to consider the mechanism of atom cooling with the help of laser radiation. Investigations in this field led to considerable progress in the understanding of the properties of quantum gases of cold atoms, and were awarded Nobel prizes in 1997 and 2001.

Theoretical Introduction

Consider a simple two-level model of the atom, with ground state energy E_g and excited state energy E_e. Energy difference is $E_g - E_e = \eta\omega_0$, the angular frequency of used laser is ω, and the laser detuning is $\delta = \omega - \omega_0 << \omega_0$. Assume that all atom velocities satisfy $\upsilon << c$, where c is the light speed. You can always restrict yourself to first nontrivial orders in small parameters υ / c and δ / ω_0. Natural width of the excited state E_e due to spontaneous decay is $\gamma << \omega_0$: for an atom in an excited state, the probability to return to a ground state per unit time equals γ. When an atom returns to a ground state, it emits a photon of a frequency close to ω_0 in a random direction.

It can be shown in quantum mechanics, that when an atom is subject to low-intensity laser radiation, the probability to excite the atom per unit time depends on the frequency of radiation in the reference frame of the atom, ω_a, according to

$$\gamma_p = s_0 \frac{\gamma / 2}{1 + 4(\omega_a - \omega_0)^2 / \gamma^2} << \gamma ,$$

where $s_0 << 1$ is a parameter, which depends on the properties of atoms and laser intensity.

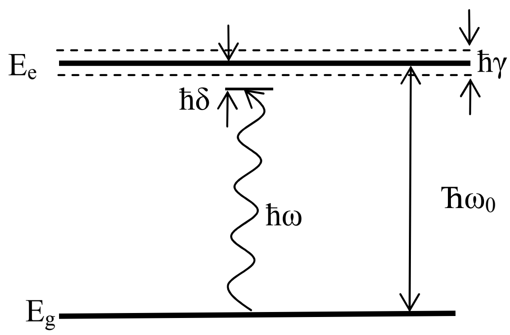

Fig 1. *Note that shown parameters are not in scale.*

In this problem properties of the gas of sodium atoms are investigated neglecting the interactions between the atoms. The laser intensity is small enough, so that the number of atoms in the excited state is always much smaller than number of atoms in the ground state. You can also neglect the effects of the gravitation, which are compensated in real experiments by an additional magnetic field.

Numerical values:

Planck constant	$\eta = 1.05 \cdot 10^{-34}$ J s
Boltzmann constant	$k_B = 1.38 \cdot 10^{-23}$ J K^{-1}
Mass of sodium atom	$m = 3.81 \cdot 10^{-26}$ kg
Frequency of used transition	$\omega_0 = 2\pi \cdot 5.08 \cdot 10^{14}$ Hz
Excited state linewidth	$\gamma = 2\pi \cdot 9.80 \cdot 10^{6}$ Hz
Concentration of the atoms	$n = 10^{14}$ cm^{-3}

Questions

a) **[1 Point]** Suppose the atom is moving in the positive x direction with the velocity υ_x, and the laser radiation with frequency ω is propagating in the negative x direction. What is the frequency of radiation in the reference frame of the atom?

b) **[2.5 Points]** Suppose the atom is moving in the positive x direction with the velocity υ_x, and two identical laser beams shine along x direction from different sides. Laser frequencies are ω, and intensity parameters are s_0. Find the expression for the average force $F(\upsilon_x)$ acting on an atom. For small υ_x this force can be written as $F(\upsilon_x) = -\beta\upsilon_x$. Find the expression for β. What is the sign of $\delta = \omega - \omega_0$, if the absolute value of the velocity of the atom decreases? Assume that momentum of an atom is much larger than the momentum of a photon.

In what follows we will assume that the atom velocity is small enough so that one can use the linear expression for the average force.

c) **[2.0 Points]** If one uses 6 lasers along x, y and z axes in positive and negative directions, then for β>0 the dissipative force acts on the atoms, and their average energy decreases. This means that the temperature of the gas, which is defined through the average energy, decreases. Using the concentration of the atoms given above, estimate numerically the temperature T_Q, for which one cannot consider atoms as point-like objects because of quantum effects.

In what follows we will assume that the temperature is much larger than T_Q and six lasers along x, y and z directions are used, as was explained in part c).

In part b) you calculated the average force acting on the atom. However, because of the quantum nature of photons, in each absorption or emission process the momentum of the atom changes by some discrete value and in random direction, due to the recoil processes.

d) **[0.5 Points]** Determine numerically the square value of the change of the momentum of the atom, $(\Delta p)^2$, as the result of one absorption or emission event.

e) **[3.5 Points]** Because of the recoil effect, average temperature of the gas after long time doesn't become an absolute zero, but reaches some finite value. The evolution of the momentum of the atom can be represented as a random walk in the momentum space with an average step $\sqrt{<\Delta p^2>}$, and a cooling due to the dissipative force. The steady-state temperature is determined by the combined effect of these two different processes. Show that the steady state temperature T_d is of the form: $T_d = \eta\gamma(x + \frac{1}{x})/(4k_B)$. Determine x. Assume that T_d is much larger than $<\Delta p^2>/(2k_B m)$.

Note: If vectors $\mathbf{P_1}$, $\mathbf{P_2}$, … , $\mathbf{P_n}$ are mutually statistically uncorrelated, mean square value of their sum is

$$<(\mathbf{P_1} + \mathbf{P_2} + \ldots + \mathbf{P_n})^2> = \mathbf{P_1}^2 + \mathbf{P_2}^2 + \ldots + \mathbf{P_n}^2$$

f) **[0.5 Point]** Find numerically the minimal possible value of the temperature due to recoil effect. For what ratio δ/γ is it achieved?

Theoretical Question 2

Oscillator damped by sliding friction

Theoretical Introduction

In mechanics, one often uses so called phase space, an imaginary space with the axes comprising of coordinates and moments (or velocities) of all the material points of the system. Points of the phase space are called imaging points. Every imaging point determines some state of the system.

When the mechanical system evolves, the corresponding imaging point follows a trajectory in the phase space which is called phase trajectory. One puts an arrow along the phase trajectory to show direction of the evolution. A set of all possible phase trajectories of a given mechanical system is called a phase portrait of the system. Analysis of this phase portrait allows one to unravel important qualitative properties of dynamics of the system, without solving equations of motion of the system in an explicit form. In many cases, the use of the phase space is the most appropriate method to solve problems in mechanics.

In this problem, we suggest you to use phase space in analyzing some mechanical systems with one degree of freedom, i.e., systems which are described by only one coordinate. In this case, the phase space is a two-dimensional plane. The phase trajectory is a curve on this plane given by a dependence of the momentum on the coordinate of the point, or vice versa, by a dependence of the coordinate of the point on the momentum.

As an example we present a phase trajectory of a free particle moving along x axis in positive direction (Fig.1).

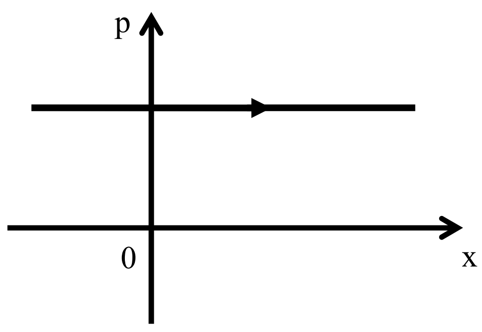

Fig. 1. Phase trajectory of a free particle.

Questions

A. Phase portraits (3.0)

A1. **[0.5 Points]** Make a draw of the phase trajectory of a free material point moving between two parallel absolutely reflective walls located at $x = - L/2$ and $x = L/2$.

A2. Investigate the phase trajectory of the harmonic oscillator, i.e., of the material point of mass m affected by Hook's force $F = - k\,x$:

a) **[0.5 Points]** Find the equation of the phase trajectory and its parameters.

b) **[0.5 Points]** Make a draw of the phase trajectory of the harmonic oscillator.

A3. **[1.5 Points]** Consider a material point of mass m on the end of weightless solid rod of length L, another end of which is fixed (strength of gravitational field is g). It is convenient to use the angle α between the rod and vertical line as a coordinate of the system. The phase plane is the plane with coordinates $(\alpha, d\alpha/dt)$. Study and make a draw of the phase portrait of this pendulum at arbitrary angle α. How many qualitatively different types of phase trajectories K does this system have? Draw at least one typical trajectory of each type. Find the conditions which determine these different types of phase trajectories. (Do not take the equilibrium points as phase trajectories). Neglect air resistance.

B. The oscillator damped by sliding friction (7.0)

When considering resistance to a motion, we usually deal with two types of friction forces. The first type is the friction force, which depends on the velocity (viscous friction), and is defined by $\boldsymbol{F} = -\gamma \boldsymbol{v}$. An example is given by a motion of a solid body in gases or liquids. The second type is the friction force, which does not depend on the magnitude of velocity. It is defined by the value $F = \mu N$ and direction opposite to the relative velocity of contacting bodies (sliding friction). An example is given by a motion of a solid body on the surface of another solid body.

As a specific example of the second type, consider a solid body on a horizontal surface at the end of a spring, another end of which is fixed. The mass of the body is m, the elasticity coefficient of the spring is k, the friction coefficient between the body and the surface is μ. Assume that the body moves along the straight line with the coordinate x ($x = 0$ corresponds to the spring which is not stretched). Assume that static and dynamical friction coefficients are the same. At initial moment the body has a position $x=A_0$ ($A_0>0$) and zero velocity.

B1. **[1.0 Points]** Write down equation of motion of the harmonic oscillator damped by the sliding friction.

B2. **[2.0 Points]** Make a draw of the phase trajectory of this oscillator and find the equilibrium points.

B3. **[1.0 Points]** Does the body completely stop at the position where the string is not stretched? If not, determine the length of the region where the body can come to a complete stop.

B4. **[2.0 Points]** Find the decrease of the maximal deviation of the oscillator in positive x direction during one oscillation ΔA. What is the time between two consequent maximal deviations in positive direction? Find the dependence of this maximal deviation $A(t_n)$ where t_n is the time of the n-th maximal deviation in positive direction.

B5. **[1.0 Points]** Make a draw of the dependence of coordinate on time, $x(t)$, and estimate the number N of oscillations of the body?

Note:

Equation of the ellipse with semi-axes a and b and centre at the origin has the following form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Theoretical Question 3

This problem consists of four not related parts.

A. [2.5 points] The Mariana Abyss in the Pacific Ocean has a depth of $H = 10920\ m$. Density of salted water at the surface of the ocean is $\rho_0 = 1025\ kg/m^3$, bulk modulus is $K = 2{,}1\cdot 10^9\ Pa$, acceleration of gravity is $g=9.81 m/s^2$. Neglect the change in the temperature and in the acceleration of gravity with the depth, and also neglect the atmospheric pressure.

A1) Find the relation between the density $\rho(x)$ and pressure $P(x)$ at the depth of x.

A2) Find the numerical value of the pressure $P(H)$ at the bottom of the Mariana Abyss. You may use iterative methods to solve this part.

Note**:** The fluids have very small compressibility. Compressibility coefficient is defined as

$$\kappa = -\frac{1}{V}\left(\frac{dV}{dP}\right)_{T=const}$$

Bulk modulus K is the inverse of κ: $K=1/\kappa$.

B. [2.5 points] Light mobile piston separates the vessel into two parts. The vessel is isolated from the environment. One part of the vessel contains $m_1 = 3g$ of hydrogen at the temperature of $T_{10} = 300$ K, and the other part contains $m_2 = 16\ g$ of oxygen at the temperature of $T_{20} = 400$ K. Molar masses of hydrogen and oxygen are $\mu_1 = 2\ g/mole$ and $\mu_2 = 32\ g/mole$ respectively, and $R = 8.31 J/(K\cdot mole)$. The piston weakly conducts heat between oxygen and hydrogen, and eventually the temperature in the system equilibrates. All the processes are quasi stationary.

B1) What is the final temperature of the system T?

B2) What is the ratio between final pressure P_f and initial pressure P_i?

B3) What is the total amount of heat Q, transferred from oxygen to hydrogen?

C. [2.5 points] Two identical conducting plates α and β with charges –Q and +q respectively (Q > q > 0) are located parallel to each other at a small distance. Another identical plate γ with mass m and charge +Q is situated parallel to the original plates at distance d from the plate β (see fig 1). Surface area of the plates is S. The plate γ is released and can move freely, while the plates α and β are kept fixed. Assume that the collision between the plates β and γ is elastic, and neglect the gravitational force and the boundary effects. Assume that the charge has enough time to redistribute between plates β and γ during the collision.

C1) What is the electric field E_1 acting on the plate γ before the collision with the plate β ?

C2) What are the charges of the plates Q_β and Q_γ after the collision?

C3) What is the velocity υ of the plate γ after the collision at the distance d from the plate β ?

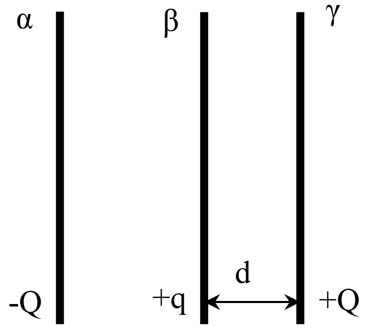

Fig. 1

D. [2.5 points] Two thin lenses with lens powers D_1 and D_2 are located at distance $L = 25cm$ from each other, and their main optical axes coincide. This system creates a direct real image of the object, located at the main optical axis closer to lens D_1, with the magnification $\Gamma' = 1$. If the positions of the two lenses are exchanged, the system again produces a direct real image, with the magnification $\Gamma'' = 4$.

D1) What are the types of the lenses? On the answer sheet you should mark the gathering lens as «+», and the diverging lens as «-» .

D2) What is the difference between the lens powers $\Delta D = D_1 - D_2$?