Theoretical Question 1

Laser Cooling of Atoms

Solutions

a) $\omega(1+\upsilon_x/c)$, this is classic Doppler effect.

b) Absolute value of the momentum, transferred during each absorption, equals

$$\eta\omega_0/c \qquad (1)$$

The momentum of the emitted photon is uniformly distributed over different directions, and after averaging gives a contribution which is much smaller than $\eta\omega_0/c$. The average force is nonzero, since for atoms moving towards right frequency of right laser gets larger(due to Doppler effect discussed in part A), while frequency of left laser goes down. Since number of scattered photons depends on the frequency in the reference frame of the atom, there is a net nonzero force. It equals

$$F(\upsilon_x)=F_+ + F_- = -(\eta\omega_0/c)\cdot(s_0\gamma/2)\cdot\left(\frac{1}{1+4(\delta+\omega_0\upsilon_x/c)^2/\gamma^2}-\frac{1}{1+4(\delta-\omega_0\upsilon_x/c)^2/\gamma^2}\right)$$

For $\upsilon_x/c << \delta/\omega_0$

$$\beta = -\frac{8\eta\omega_0^2\delta s_0}{\gamma c^2(1+4(\delta/\gamma)^2)^2}$$

For β>0, one needs

$$\delta<0.$$

c) Characteristic de-Broglie wavelength at temperature T equals $\lambda=\eta/\sqrt{mk_BT}$. To consider the atoms as point-like objects one needs this distance to be much smaller than characteristic inter particle separation $n^{-1/3}$. From the condition that these two lengths are of the same order of magnitude we get

$$T_Q=(\eta^2 n^{2/3})/(k_B m)\approx 10^{-6}K$$

d) $<\Delta p^2>=\eta^2\omega_0^2/c^2\approx 10^{-54}kg^2m^2/s^2$ - this is the mean square recoil momentum of a photon.

e) Assume that the steady state value of the average square of the momentum of atom equals P_0^2. In steady state regime this quantity doesn't change with time, and temperature is obtained according to $3k_BT_d/2=P_0^2/(2m)$. Let the momentum at some point of time in steady state regime be P_0. Let's consider the value of the momentum after some time *t*. During this time the atom will participate in $N = 6\gamma_p t >> 1$ absorption-emission processes (6 comes from the number of lasers). For each absorption-emission event the atom gets two recoil kicks, each with a mean square value $<\Delta p^2>$ calculated in part d) (one kick is during absorption and one is during emission). The directions of these kicks are uncorrelated for different events, so this leads to an increase of the mean square of the momentum by $2N<\Delta p^2>$.

On the other hand, atoms are cooled because of the dissipative force, and the change of the mean square of the momentum because of this process is $-2\beta P_0^2 t/m$. For steady state solution these two processes compensate each other, so we obtain:

$$P_0^2 = 12<\Delta p^2>\gamma_p m/(2\beta) = 3\eta m\gamma\left(\frac{2|\delta|}{\gamma}+\frac{\gamma}{2|\delta|}\right)/4$$

Thus the temperature

$$T_d = \eta\gamma\left(\frac{2|\delta|}{\gamma}+\frac{\gamma}{2|\delta|}\right)/(4k_B)$$

f) The minimum is achieved for $\delta = -\gamma/2$, and equals $\eta\gamma/(2k_B) = 2.4\cdot 10^{-4}\,K$

Mark Distribution

No	Total Pt	Partial Pt	Contents
a)	1pt		Expression for Doppler shifted frequency
b)	2.5	0.3	Expression for photon momentum (1)
		0.5	Correct F_- or F_+ with the Doppler shifted frequency
		1.0	Expression for $F(v_x)$
		0.5	Expression for β
		0.2	The correct sign of δ
c)	2.0	0.5	Expression for characteristic de Broglie wavelength or an estimate from uncertainty principle
		0.5	Expression for characteristic inter particle separation
		0.5	The equality between De Broglie wavelength and inter-particle separation
		0.5	Numerical value of temperature
d)	0.5		Numerical answer
e)	3.5	0.5	Calculation of the number of absorption-emission processes during time t
		1.5	Expression for the change of the mean square momentum (or energy) for time t due to random kicks (heating rate)
		0.5	Expression for the cooling rate
		1.0	Final expression x
f)	0.5	0.3	Expression for minimal temperature and numerical value
		0.2	$\delta = -\gamma / 2$

Theoretical Question 2

Solution

A. Phase portraits (3.0)

A1. **[0.5 p]** Let Ox axis be pointed perpendicular to the walls. Since the material point is free and collisions are absolutely elastic then the magnitude of momentum is conserved, while its direction is changed to opposite at the collisions. Hence, the phase trajectory is of the following form (Fig. 1):

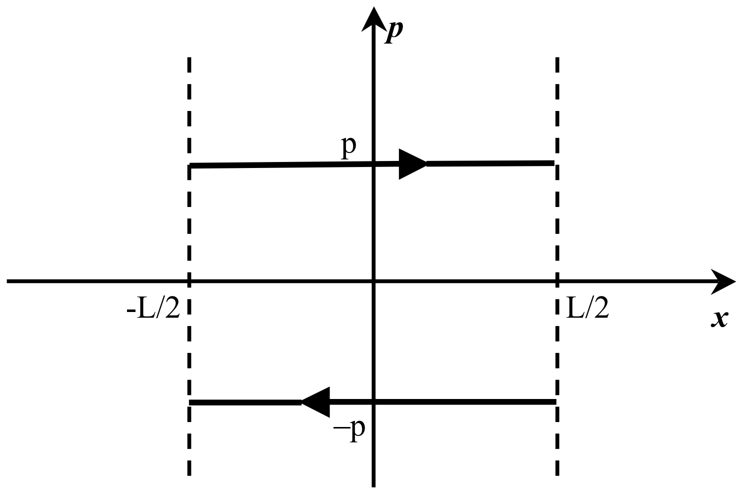


Fig. 1.

The motion with positive values of the momentum is directed along increasing values of the coordinate. Thus, the phase trajectory is directed clockwise, as indicated in Fig. 1.

A2. **[1.0 p]**

a) **[0.5 p]** For the harmonic oscillator, let us denote the coordinate by x, the momentum by p, and the total energy by E. The energy conservation law is

$$\frac{p^2}{2m}+\frac{kx^2}{2}=E$$

This expression determines the equation of phase trajectory, for a given E. Dividing both sides of the equation by E, we obtain

$$\frac{p^2}{2mE}+\frac{x^2}{2E/k}=1$$

This is a canonical form of the equation of ellipse in (x, p). The centre of the ellipse is at (0, 0) and the semiaxes are $\sqrt{2E/k}$ and $\sqrt{2Em}$ respectively.

b) **[0.5 p]** The phase trajectory is of the following form (Fig. 2):

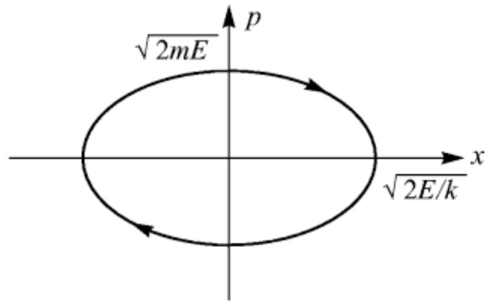


Fig. 2.

The motion with positive values of the momentum is directed along increasing values of the coordinate. Thus, the phase trajectory is directed clockwise, as indicated in Fig. 1.

A3. **[1.5 p]** Let us choose the potential energy level at the lowest point of the pendulum (equilibrium state). Taking into account for that linear velocity of the point is $v=L\dot{\alpha}$, we write down the total energy of the mathematical pendulum

$$\frac{mL^2\dot{\alpha}^2}{2}+mgL(1-\cos\alpha)=E$$

Analysis of this expression leads to the following:

at $E<2mgL$ the pendulum oscillates about the lower equilibrium position; if $E<<mgL$ then the oscillations are harmonic;

at $E=2mgL$ the pendulum does not oscillate; the pendulum tends to the upper point of equilibrium.

at $E>2mgL$ the pendulum rotates about fixed point.

The phase trajectory is shown in Fig. 3.

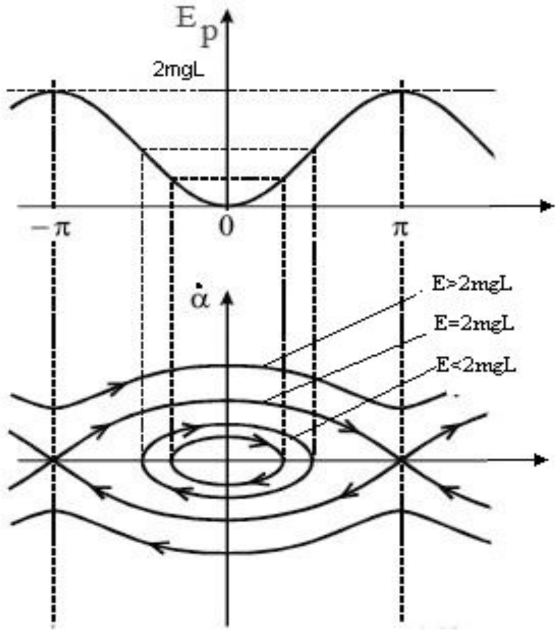


Fig. 3. (0.5)

There are K = 3 qualitatively different types of the phase trajectories: oscillations, rotations, and the motion to the upper point of equilibrium (separatrisse) (1.0). (We do not take the equilibrium points as phase trajectories)

B. The oscillator damped by sliding friction. (7.0)

B1. **[1.0 p]** For the sliding friction the magnitude of the friction force does not depend on the magnitude of velocity, but its direction is opposite to the velocity vector of the body. Therefore, the equations of motion should be written separately for the motion to the right and to the left from the "equilibrium" point (the spring is not stretched). Let us choose the *x*-axis along the direction of motion, and the origin of the coordinate system at the equilibrium point without the friction force. We obtain the equations of motion as follows:

$$\begin{aligned} \ddot{x} + {\omega_0}^2 x &= -\frac{F_{fr}}{m}, \quad \dot{x} > 0 \\ \ddot{x} + {\omega_0}^2 x &= \frac{F_{fr}}{m}, \quad \dot{x} < 0 \end{aligned} \qquad (1)$$

Here, $F_{fr} = \mu mg$ is the friction force, ${\omega_0}^2 = k/m$ is the frequency of oscillations of the pendulum without the friction.

B2. **[2.0 p]** Introducing the variables $x_1 = x + F_{fr}/m\omega_0^2$ and $x_2 = x - F_{fr}/m\omega_0^2$ we can write down the equations of motion in the same form for both the cases,

$$\ddot{x}_{1,2} + {\omega_0}^2 x_{1,2} = 0 \ (0.5)$$

which coincides with the equation of motion of harmonic oscillator without the friction. The action of the friction force is reduced to a drift of the equilibrium points: for $\dot{x} > 0$, it becomes $x_- = -F_{fr}/m\omega_0^2$, $x_1 = 0$ and for $\dot{x} < 0$ it becomes $x_+ = F_{fr}/m\omega_0^2$, $x_2 = 0$ (0.5).

Thus, due to the section A 2 above the phase trajectory is a combination of parts of ellipses with centers at the point x_- for an upper half-plane p>0, and at the point x_+ for a lower half-plane p<0. As the result of a continuity of motion these parts of ellipses should comprise a continuous curve by meeting each other at p = 0.

Thus, the phase trajectory is (Fig. 4) (2.5)

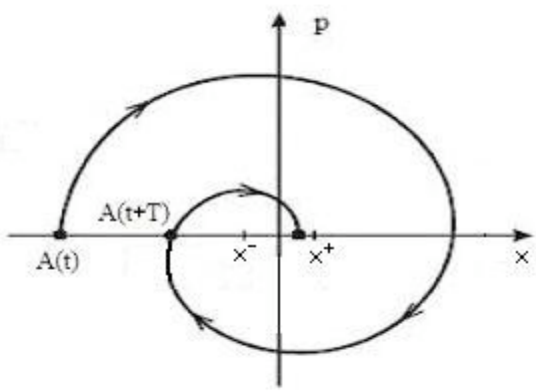


Fig. 4. (1.0)

B3. **[1.0 p]** According to the phase trajectory combination, the body not necessarily stops at the point x=0. It will stop when it falls into the range from x_- to x_+. (0.5) This region is called stagnation region. The width of this region is

$$x_+ - x_- = \frac{2F_{fr}}{m\omega_0^2} \quad (0.5)$$

B4. **[1.5 p]** From the definition of equilibrium points and the obtained form of phase trajectory it is easy to find the decrease of amplitude during one period:

$$\Delta A = A(t) - A(t+T) = 2(x_+ - x_-) = \frac{4F_{fr}}{m\omega_0^2} \quad (0.5)$$

This can be rewritten as

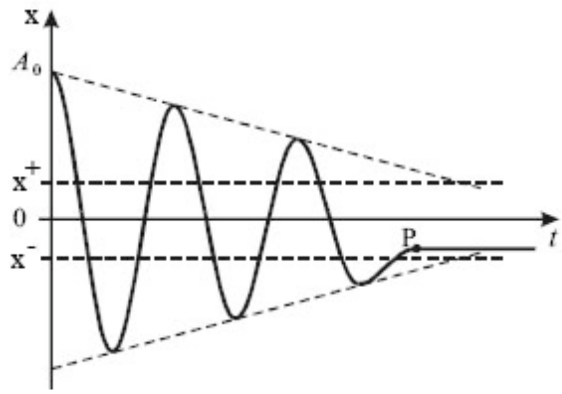
$$A(t) - A(t+T) = \frac{4F_{fr}}{2\pi m\omega_0}T \quad (0.5)$$

One can see that, unlike the case of viscous friction, the amplitude decreases in accord to a linear law, $A = A_0 - pt_n$, where $p = {}^{2F_{fr}}\!/_{\pi m\omega_0}$ (0.5).

B5. **[1.5 p]** The total number of oscillations depends on the initial amplitude A_0, and it can be found as

$$N = \frac{A_0}{2(x_+ - x_-)} \quad (0.5)$$

As the result of the above conclusions the plot of $x(t)$ is of the following form (Fig. 5)



(0.5)

Fig. 5.

The frequency is equal to the frequency of free oscillator, $\omega_0^2 = k/m$. The time between two successive maximal deviations is $T=2\pi/\omega_0$ (0.5)

The oscillations do not stop until the amplitude is more than half-width of the stagnation region $x_+ - x_-$. In real situations, the body stops in random positions within the stagnation region. In Fig. 5 the point P denotes the point where the body stops.

Marking scheme of Theoretical Question 2

	A. Phase portraits (3.0)	
A1 [0.5]	The draw of phase trajectory with the arrows indicating the direction of motion.	**0.5 Points**
A2 [1.0]	**a)** $\frac{p^2}{2mE}+\frac{x^2}{2E/k}=1$ with semiaxes $\sqrt{2E/k}$ and $\sqrt{2Em}$	**0.5 Points**
	b) The draw of the ellipse with the arrows indicating the direction of motion.	**0.5 Points**
A3 [1.5]	$K=3$ Discussion of three cases: $E<2mgL$, $E=2mgL$ and $E>2mgL$.	**1.0 Point**
	Phase portrait with the arrows indicating the direction of motion.	**0.5 Points**
	B. The oscillator damped by sliding friction (7.0)	
B1 [1.0]	$\ddot{x}+\omega_0{}^2x=-\frac{F_{fr}}{m},\quad \dot{x}>0$ $\ddot{x}+\omega_0{}^2x=\frac{F_{fr}}{m},\quad \dot{x}<0$	**1.0 Point**
B2 [2.0]	$\ddot{x}_{1,2}+\omega_0{}^2x_{1,2}=0$	**0.5 Points**
	Equilibrium points $x_-=-F_{fr}/m\omega_0^2$, $x_1=0$, and $x_+=F_{fr}/m\omega_0^2$, $x_2=0$	**0.5 Points**
	Phase trajectory with equilibrium points and the arrows indicating the direction of motion	**1.0 Point**
B3 [1.0]	Complete stop within (x_-,x_+)	**0.5 Points**
	$x_+-x_-=\frac{2F_{fr}}{m\omega_0^2}$	**0.5 Points**
B4 [2.0]	$\Delta A=A(t)-A(t+T)=2(x_+-x_-)=\frac{4F_{fr}}{m\omega_0^2}$	**0.5 Points**
	$A(t)-A(t+T)=\frac{4F_{fr}}{2\pi m\omega_0}T$ or analogous expression	**0.5 Points**
	$A(t)=A_0-pt$, where $p=\frac{2F_{fr}}{\pi m\omega_0}$	**0.5 Points**
	t_n-t_{n-1}	**0.5 Points**
B5 [1.0]	$N=\frac{A_0}{2(x_+-x_-)}$	**0.5 Points**
	The draw of x(t) with a linear decrease of amplitude and complete stop.	**0.5 Points**

Theoretical Question 3

Solutions

A

The change of the pressure is related to the change in the density via

$$\Delta P = -K\frac{\Delta V}{V} = K\frac{\Delta\rho}{\rho_0}, \qquad (1)$$

where ρ_0 is the density of water at the surface.

$$\rho = \rho_0 + \Delta\rho = \rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right) = \rho_0\left(1+\frac{\Delta P}{K}\right), \qquad (2)$$

where $\Delta P \approx P$ (we neglect the atmospheric pressure). Then

(A1)
$$\rho(x) = \rho_0\left(1+\frac{P(x)}{K}\right). \qquad (3)$$

The change of the hydrostatic pressure with the depth equals

$$dP = g\cdot\rho(x)dx, \quad \frac{dP}{dx} = g\rho(x) = g\rho_0 + g\rho_0\frac{P(x)}{K}, \qquad (4)$$

$$\frac{dP(x)}{dx} - \frac{g\rho_0}{K}P(x) = g\rho_0. \qquad (5)$$

The solution of this differential equation with boundary condition $P(0)=0$ is

$$P(x) = K\left(\exp\frac{g\rho_0}{K}x - 1\right). \qquad (6)$$

Since $\frac{g\rho_0}{K}H << 1$, we can use the expansion

$$\exp z \approx 1 + z + \frac{z^2}{2!} + \ldots, \qquad (7)$$

thus

$$P(x) \cong g\rho_0 x + \frac{1}{2K}(g\rho_0 x)^2 \qquad (8)$$

The last formula can be simply derived using the method of successive iterations. First, the pressure can be estimated without compressibility taken into account:

$$P_0(x) = g\rho_0 x. \qquad (9)$$

Correction to the density in the first approximation can be obtained using $P_0(x)$:

$$\rho_1(x) = \rho_o(1+\frac{g\rho_0 x}{K}). \qquad (10)$$

Now, correction to pressure can be obtained using $\rho_1(x)$:

$$P_1(H) = \int_0^H \rho_1(x) g dx = g\rho_0 x + \frac{1}{2K}(g\rho_0 x)^2, \qquad (11)$$

as obtained earlier.

Putting in the numerical values, we get

(A2) $$P(H) = (1098\cdot 10^5 + 28{,}7\cdot 10^5)\Pi a \approx 1{,}13\cdot 10^8\ \Pi a. \qquad (12)$$

B

The total work done by the gases is zero. Thus at any moment the total internal energy equals the original value:

$$\frac{m_1}{\mu_1}C_V T_1 + \frac{m_2}{\mu_2}C_V T_2 = \frac{m_1}{\mu_1}C_V T_{10} + \frac{m_2}{\mu_2}C_V T_{20,} \qquad (13)$$

where $\mu_1 = 2\ g/mole$ and $\mu_2 = 32\ g/mole$ are molar masses of hydrogen and oxygen, and $C_V=5R/2$ is the molar heat capacity of diatomic gas. The final temperature of the system is

(B1) $$T = \frac{\frac{m_1}{\mu_1}T_{10} + \frac{m_2}{\mu_2}T_{20}}{\frac{m_1}{\mu_1} + \frac{m_2}{\mu_2}} = 325\ K. \qquad (14)$$

The temperature of oxygen decreases, and the amount of heat Q is transferred to hydrogen by heat conduction. The piston will move in the direction of the oxygen, thus the hydrogen does a positive work *A>0*, and the change of the internal energy of oxygen is $\Delta U = A - Q$. On the other hand,

$$\Delta U = \frac{m_2}{\mu_2}\frac{5}{2}R(T - T_{20}) = -779\ J. \qquad (15)$$

To find *A,* let us prove that the pressure *P* doesn't change. Differentiating the equations of the state for each gas, we get

$$\Delta T_1 = \frac{\mu_1}{m_1 R}(P\Delta V + V_1\Delta P), \quad \Delta T_2 = \frac{\mu_2}{m_2 R}(-P\Delta V + V_2\Delta P). \qquad (16)$$

where V_i are the gas volumes, and $\Delta V = \Delta V_1 = -\Delta V_2$ is the change of the volume of the hydrogen. Differentiating (1), we get

$$\frac{m_1}{\mu_1}\Delta T_1 + \frac{m_2}{\mu_2}\Delta T_2 = 0. \tag{17}$$

Substituting (2) into (3), we obtain $(V_1+V_2)\cdot\Delta P = 0$, thus

(B2) $$P_f / P_i = 1. \tag{18}$$

Then the work done by the hydrogen is

$$A = P\cdot\Delta V = -\frac{m_2}{\mu_2}R\cdot\Delta T_2 = \frac{m_2}{\mu_2}R(T_{20}-T) = 312\ J. \tag{19}$$

The total amount of heat transferred to hydrogen is

(B3) $$Q = A - \Delta U = 1091 J. \tag{20}$$

C

The electric field acting on the plate γ before the collision is

(C1) $$E_1 = \frac{Q-q}{2\varepsilon_0 S}. \tag{21}$$

The force acting on the plate is

$$F_1 = E_1 Q = \frac{(Q-q)Q}{2\varepsilon_0 S}. \tag{22}$$

The work done by the electric field before the collision is

$$A_1 = F_1 d = \frac{(Q-q)Qd}{2\varepsilon_0 S}. \tag{23}$$

The charge will get redistributed between two touching conducting plates during the collision. The values of the charges can be obtained from the condition that the electric field between the touching plates vanishes. If one assumes that the plate γ is on the right side, the left surface of the combined plate will have the charge

(C2a) $$Q_\beta = Q + q/2, \tag{24}$$

and the right surface will have the charge

(C2b) $$Q_\gamma = q/2. \tag{25}$$

These charges remain on the plates after the collision is over. Now the force acting on the plate γ equals $F_2 = E_2 q/2$, where $E_2 = (q/2)/(2\varepsilon_0 S)$. The work done by field E_2 is

$$A_2 = F_2 d = \frac{q^2 d}{8\varepsilon_0 S}. \tag{26}$$

The total work done by the electric fields is

$$A = A_1 + A_2 = \frac{d}{2\varepsilon_0 S}\left(Q - \frac{q}{2}\right)^2. \tag{27}$$

Velocity at the distance d can be calculated using the following relation:

$$\frac{m\upsilon^2}{2} = A. \tag{28}$$

Substituting (5) into (6), we finally get

(C3) $$\upsilon = \left(Q - \frac{q}{2}\right)\sqrt{\frac{d}{m\varepsilon_0 S}}. \tag{29}$$

D

First one has to determine the types of the lenses. If both lenses are negative, one always obtains a direct imaginary image. If one lens is positive and the other is negative, three variants are possible: an inversed real image, a direct imaginary image or an inversed imaginary image, all contradicting the conditions of the problem. Only the last variant is left – two positive lenses. The first lens creates an inversed real image, and the second one inverts in once more, creating the direct real image. Using the lens equations, the magnifications of the lenses can be written as

$$\Gamma_1 = \frac{F_1}{d_1 - F_1}; \qquad \Gamma_2 = \frac{F_2}{d_2 - F_2}, \tag{30}$$

where d_1 is the distance from the object to the first lens, $d_2 = L - f_1$ is the distance from the image of the first lens to the second lens, and f_1 is the distance from the first lens to the first image. The total magnification of the system is $\Gamma' = \Gamma_1 \cdot \Gamma_2$. Using the expression for d_2, inverted magnification coefficient can be written as

$$\frac{1}{\Gamma'} = \frac{d_1\left[L - (F_1 + F_2)\right]}{F_1 F_2} - \frac{L}{F_2} + 1. \tag{31}$$

One notices from this expression that if two lenses are exchanged, the first term stays invariant, and only the second term changes. Thus the expression for the inverted magnification in the second case is:

$$\frac{1}{\Gamma''} = \frac{d_1\left[L-(F_1+F_2)\right]}{F_1F_2} - \frac{L}{F_1} + 1. \qquad (32)$$

Subtracting these two formulas, we get:

$$\frac{1}{\Gamma'} - \frac{1}{\Gamma''} = L\left(\frac{1}{F_1} - \frac{1}{F_2}\right) = L(D_1 - D_2); \qquad (33)$$

$$D_1 - D_2 = \frac{1}{L}\left(\frac{1}{\Gamma'} - \frac{1}{\Gamma''}\right) = \frac{1}{0,25}\left(1 - \frac{1}{4}\right) = \frac{1}{0,25}\cdot\frac{3}{4} = 3 \text{ diopters.} \qquad (34)$$

Mark Distribution

<table>
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<tr><td rowspan="8">A</td><td rowspan="8">2.5</td><td>0.5</td><td colspan="3">Equation</td></tr>
<tr><td rowspan="5">1.5</td><td rowspan="2">Differential equation method</td><td>1.0</td><td>Equation (4) or (5)</td></tr>
<tr><td>0.5</td><td>Equation (6)</td></tr>
<tr><td rowspan="3">Iterative method</td><td>0.5</td><td>$P_0(x)$</td></tr>
<tr><td>0.5</td><td>$\rho_1(x)$</td></tr>
<tr><td>0.5</td><td>$P_1(H)$</td></tr>
<tr><td>0.5</td><td colspan="3">Numerical value of P(H)</td></tr>
<tr><td rowspan="5">B</td><td rowspan="5">2.5</td><td>0.5</td><td colspan="3">Numerical value of T</td></tr>
<tr><td>1.0</td><td colspan="3">Constant pressure</td></tr>
<tr><td>0.3</td><td colspan="3">Numerical value of ΔU</td></tr>
<tr><td>0.3</td><td colspan="3">Numerical value of A</td></tr>
<tr><td>0.4</td><td colspan="3">Numerical value of Q</td></tr>
<tr><td rowspan="5">C</td><td rowspan="5">2.5</td><td>0.5</td><td colspan="3">E_1</td></tr>
<tr><td>0.5</td><td colspan="3">Q_β</td></tr>
<tr><td>0.5</td><td colspan="3">Q_γ</td></tr>
<tr><td>0.5</td><td colspan="3">Total work by electric fields</td></tr>
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<tr><td rowspan="3">D</td><td rowspan="3">2.5</td><td>1.0</td><td colspan="3">Signs of lenses</td></tr>
<tr><td>0.5</td><td colspan="3">Expression for the magnification coefficient</td></tr>
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