Solution of the theoretical problem 1

Back-and-Forth Rolling of a Liquid-Filled Sphere

1.（1） Let I_1 and I_2 denote the rotational inertia of the spherical shell and **W** in solid state respectively, while I be the sum of I_1 and I_2 . The surface mass density of the spherical shell is $\sigma = \frac{m}{4\pi r^2}$. Cut a narrow zone from the spherical shell perpendicular to its diameter, which spans a small angle $\mathrm{d}\alpha$ with respect to the center of the sphere C, while the spherical zone makes an angle α with the diameter of the spherical shell, which is called C axes hereafter, as shown in Fig. 1. The rotational inertia of the narrow zone about the C axis is $2\pi r \sin\alpha (r\mathrm{d}\alpha)\sigma(r\sin\alpha)^2$, therefore integral over the whole spherical shell gives

$$I_1 = \int_0^{\pi} 2\pi r \sin\alpha (r\mathrm{d}\alpha)\sigma(r\sin\alpha)^2 = \frac{2}{3}mr^2. \tag{1A.1}$$

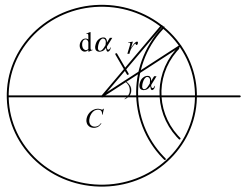


Figure 1

The volume density of **W** is $\rho = \frac{M}{4\pi r^3/3}$. By using above result for the spherical zone it can be seen that the rotational inertia of the solid **W** about the C axis is

$$I_2 = \int_0^{r} \frac{2}{3} r'^2 \cdot \rho \cdot 4\pi r'^2 \,\mathrm{d}r' = \frac{2}{5}Mr^2. \tag{1A.2}$$

Then,
$$I = I_1 + I_2 = \frac{2}{3}mr^2 + \frac{2}{5}Mr^2. \tag{1A.3}$$

（2）According to the Newton’s second law we can derive the translational motion equation of the center of mass for the sphere along the tangent of the bowl,

$$(m+M)(R-r)\ddot{\theta} = -(m+M)g\theta + f, \tag{1A.4}$$

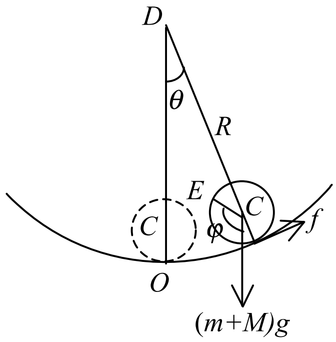


Figure 2

where $\theta(\theta << 1)$ denotes the angular position of the center of mass of the sphere as shown in Fig.2, and f is the frictional force acting on the sphere by the inside wall of the bowl. From the rotational dynamics, we have,

$$f\,r = -I\ddot{\varphi} = -(\frac{2}{3}\;mr^2 + \frac{2}{5}\;Mr^2)\ddot{\varphi}, \tag{1A.5}$$

where φ is the angular position of the reference radius CE with respect to the starting position. Assumed constraint of pure rolling on the motion of the sphere reads,

$$(R-r)\ddot{\theta} = r\ddot{\varphi}, \tag{1A.6}$$

Equations (1A.4)-(1A.6) lead to

$$(\frac{5}{3}m + \frac{7}{5}M)(R-r)\ddot{\theta} = -(m+M)g\theta.$$

This is a motion equation of the type of simple harmonic oscillator. Therefore, we obtain the angular frequency and period of the sphere rolling right and left:

$$\omega_1 = \sqrt{\frac{m+M}{5m/3+7M/5}\frac{g}{R-r}}\;, \tag{1A.7}$$

$$T_1 = 2\pi\sqrt{\frac{R-r}{g}}\sqrt{\frac{5m/3+7M/5}{m+M}}\;. \tag{1A.8}$$

2. This case can be treated similarly, except taking that the ideal liquid does not rotate into consideration. Therefore Eqs. (1A.4) and (1A.6) are still applicable, while Eq. (1A.5) needs to be modified as

$$f\,r = -I_1\ddot{\varphi} = -\frac{2}{3}\;mr^2\ddot{\varphi}. \tag{1A.9}$$

Equations (1A.4), (1A.6), and (1A.9) result in

$$(5m/3+M)(R-r)\ddot{\theta} = -(m+M)g\theta.$$

Then, the angular frequency and period of the sphere rolling back-and-forth are obtained respectively.

$$\omega_2 = \sqrt{\frac{m+M}{5m/3+M}}\sqrt{\frac{g}{R-r}}, \tag{1A.10}$$

$$T_2 = 2\pi\sqrt{\frac{R-r}{g}}\sqrt{\frac{5m/3+M}{m+M}}. \tag{1A.11}$$

3. The time taken by the sphere from position A_0 to equilibrium position O is $T_2/4$, $T_1/4$ from O to $A_0^{'}$, and $T_2/4$ from $A_0^{'}$ to O, $T_1/4$ from O to A_1. Although the angular amplitude decreases step by step (see below) during the rolling process of the sphere right and left, the period keeps unchanged. This means

$$T_3 = \frac{1}{2}(T_1+T_2) = \pi\sqrt{\frac{R-r}{g}}\left(\sqrt{\frac{5m/3+7M/5}{m+M}}+\sqrt{\frac{5m/3+M}{m+M}}\right). \tag{1A.12}$$

Next, we calculate the change of the angular amplitude. When the sphere passes through the equilibrium position O after it rolled down from the initial position A_0, the velocity of its center is

$$v_C = \omega_2(R-r)\theta_0 = \sqrt{\frac{m+M}{5m/3+M}}\sqrt{g(R-r)}\theta_0. \tag{1A.13}$$

Now the angular velocity of the spherical shell rotating about the C axis is

$$\Omega = \frac{v_C}{r} = \frac{\theta_0}{r}\sqrt{\frac{m+M}{5m/3+M}}\sqrt{g(R-r)}. \tag{1A.14}$$

where C axis is the axis of rotation through the center of the sphere and perpendicular to the paper plane of Fig.2. When **W** behaves as liquid (before it changes into solid state), the angular momentum of the sphere relative to point O is

$$L = (m+M)v_C r + I_1\Omega. \tag{1A.15}$$

When **W** changes suddenly into solid state, due to the fact that both gravitational and frictional force pass through point O, the angular momentum of the sphere relative to O is conserved, we have

$$L = (m+M)v_C r + I_1\Omega = [I+(m+M)r^2]\Omega'. \tag{1A.16}$$

where Ω and Ω' represent the angular velocity of the sphere immediately before and after passing through point *O*. Therefore

$$\Omega' = \frac{(m+M)v_C r + I_1\Omega}{I+(m+M)r^2} = \frac{v_C}{r}\frac{5m/3+M}{5m/3+7M/5}, \tag{1A.17}$$

while after passing through point *O* the velocity of the center of the sphere becomes

$$v_C^{'} = \Omega' r = v_C \frac{5m/3+M}{5m/3+7M/5}. \tag{1A.18}$$

Once the sphere reaches the left highest position $A_0^{'}$ corresponding to the left angular amplitude $\theta_0^{'}$ we have

$$v_C' = \omega_1 (R-r)\theta_0'.$$

However, $$v_C = \omega_2 (R-r)\theta_0.$$

From above two expressions we obtain

$$\theta_0^{'} = \frac{v_c^{'}\omega_2}{v_C\omega_1}\theta_0 = \theta_0\sqrt{\frac{5m/3+M}{5m/3+7M/5}}. \tag{1A.19}$$

Similarly we can treat the process that the sphere rolls from position $A_0^{'}$ back to A_1, the second highest position on the right, corresponding to the second right angular amplitude θ_1, and obtain

$$\frac{\theta_1}{\theta_0'} = \frac{\theta_0'}{\theta_0}.$$

Then, $$\theta_1 = \frac{\theta_0^{'2}}{\theta_0} = \frac{5m/3+M}{5m/3+7M/5}\theta_0.$$

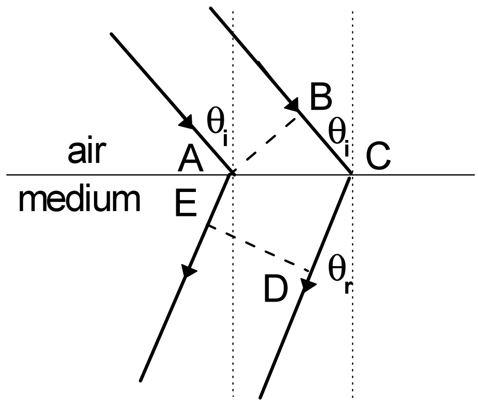
Following the similar procedure repeatedly we finally obtain:

$$\theta_n = \left(\frac{5m/3+M}{5m/3+7M/5}\right)^n \theta_0. \tag{1A.20}$$

Solution of theoretical problem 2

2A. Optical properties of an unusual material

1. （1）



Prove：Assume E-D shows one of the wavefronts of the refracted light. According to the Huygens' principle the phase accumulation from A to E should be equal to that from B via C to D:

$$\phi_{AE} = \phi_{BC} + \phi_{CD} \qquad (2A.1)$$

With the Hints given these phase differences can be calculated respectively, then

$$-\sqrt{\varepsilon_r \mu_r} k d_{AE} = k d_{BC} - \sqrt{\varepsilon_r \mu_r} k d_{CD} \qquad (2A..2)$$

Simplification of （2A.2）gives

$$-\sqrt{\varepsilon_r \mu_r}\left(d_{AE} - d_{CD}\right) = d_{BC} \qquad (2A.3)$$

Because $d_{BC} > 0$ and $\sqrt{\varepsilon_r \mu_r} > 0$， we obtain

$$d_{AE} < d_{CD} \qquad (2A.4)$$

Therefore the schematic ray diagram of the refracted light shown in above figure is reasonable.

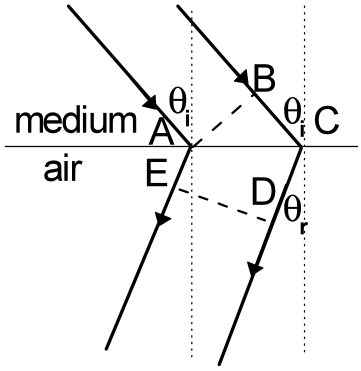
（2）From the above figure，the refraction angle θ_r and incidence angle θ_i satisfy

$$d_{BC} = d_{AC}\sin\theta_i，\quad d_{CD} - d_{AE} = d_{AC}\sin\theta_r \qquad (2A.5)$$

respectively. Substitution of（2A.5）into（2A.3）results in：

$$\sqrt{\varepsilon_r \mu_r}\sin\theta_r = \sin\theta_i \qquad (2A.6)$$

(3)



Prove: Assume E-D shows one of the wavefronts of refracted light.

According to the Huygens' principle the phase accumulation from A to E should be equal to that from B via C to D:

$$\phi_{AE} = \phi_{BC} + \phi_{CD} \tag{2A.7}$$

With the Hints given these phase differences can be calculated respectively, then

$$kd_{AE} = -\sqrt{\varepsilon_r \mu_r}\, kd_{BC} + kd_{CD} \tag{2A.8}$$

Simplification of (2A.8) gives

$$d_{AE} - d_{CD} = -\sqrt{\varepsilon_r \mu_r}\, d_{BC} \tag{2A.9}$$

Because $d_{BC} > 0$ and $\sqrt{\varepsilon_r \mu_r} > 0$, we obtain

$$d_{AE} < d_{CD} \tag{2A.10}$$

Therefore the schematic ray diagram of the refracted light shown in the above figure is reasonable.

(4) From the above figure, the refraction angle θ_r and incidence angle θ_i satisfy

$$d_{BC} = d_{AC} \sin\theta_i, \quad d_{CD} - d_{AE} = d_{AC} \sin\theta_r \tag{2A.11}$$

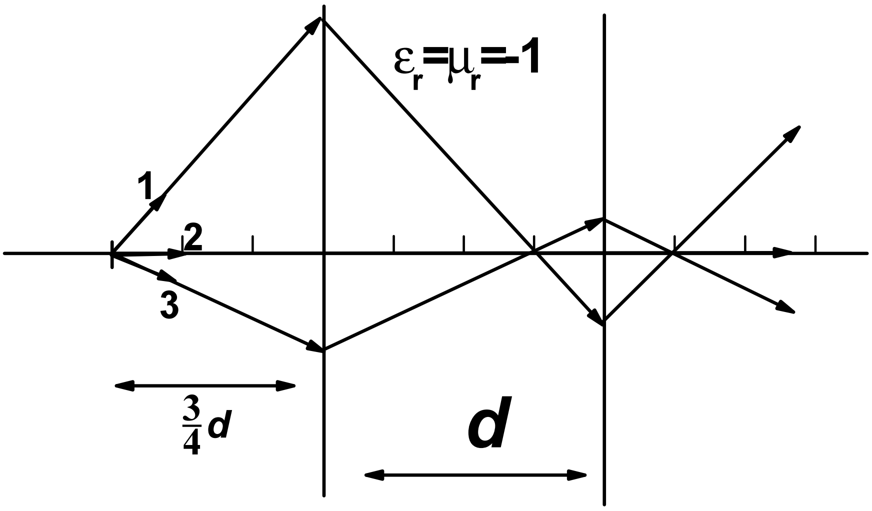
respectively. Substitution of (2A.11) into (2A.9) results in :

$$\sin\theta_r = \sqrt{\varepsilon_r \mu_r} \sin\theta_i \tag{2A.12}$$

2. The ray diagram is shown below.

Illustration:

The light is negatively refracted at both interfaces, and the refraction angle equals to incidence angle. Meanwhile according to the Hints provided there is no reflected light from each interface. Therefore within the medium light rays converge strictly at a point symmetric to the source about the left side of the medium, and on the other side of the medium the rays converge strictly at a point which is symmetric to the image of the source within the medium about the right side of the medium.



3. The phase difference between the two waves transmitting through the right side of the medium in succession is

$$\Delta\phi = 2k(d-0.4d) - 2\cdot 0.5k\cdot 0.4d + 2\pi \qquad \text{(2A.13)}$$

On the right side of above equation the first term shows the phase difference of the light wave accumulated during its propagation in air, the second term shows the phase difference of the light wave accumulated during its propagation in the unusual medium, while the third term accounts for the phase difference of the light wave accumulated due to the two reflections in succession from the interface between air and the medium. Taking $k = \frac{2\pi}{\lambda}$, (2A.13) changes into

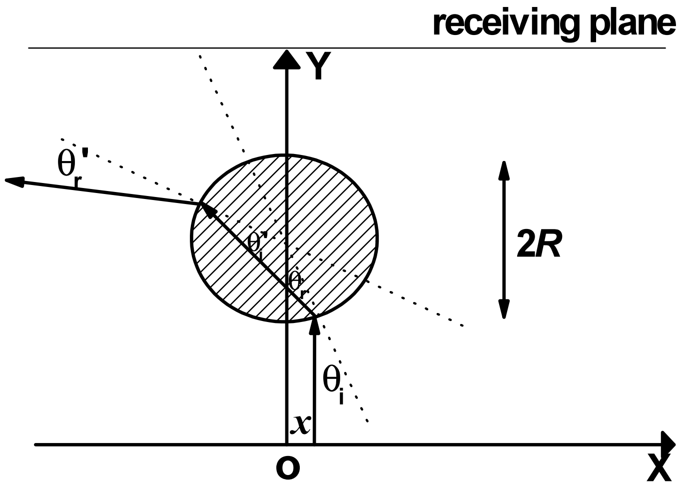
$$\Delta\phi = 0.8\frac{2\pi}{\lambda}d + 2\pi \qquad \text{(2A.14)}$$

Resonant condition means

$$0.8\frac{2\pi}{\lambda}d + 2\pi = m\cdot 2\pi \qquad \text{(2A.15)}$$

Thus $$\lambda = \frac{0.8d}{m-1}, \qquad m = 2,3,4,... \qquad \text{(2A.16)}$$

4.



From the given conditions the ray diagram can be accordingly constructed. Above figure shows schematically the ray diagram for $x > 0$.Because for the unusual medium $\varepsilon_r = \mu_r = -1$, from the solution of the first question we have $\theta_i = \theta_r = \theta_i' = \theta_r'$.Therefore the direction of the final out-going light deviates from that of the incident light by $4\theta_i$.Because the direction of the incident light is given in the y direction, only if the condition

$$\pi/2 \le 4\theta_i \le 3\pi/2 \quad \Rightarrow \quad \pi/8 \le \theta_i \le 3\pi/8 \tag{2A.17}$$

is satisfied the light signal can not reach the receiving plane. Notice

$$\sin\theta_i = x/R \text{ ,} \tag{2A.18}$$

and the similarity of the monotonicity of $\sin\theta$ to that of θ in the range of $[0,\pi/2]$, we find that（2A. 17）goes to

$$\sin(\pi/8) \le \frac{x}{R} \le \sin(3\pi/8) \quad . \tag{2A.19}$$

Further taking the symmetry about the y axis into consideration we obtain that if the following condition

$$R\sin(\pi/8) \le |x| \le R\sin(3\pi/8) \tag{2A.20}$$

is satisfied, the light emitted from a light source located on the x axis can not reach the receiving plane.

2B. Dielectric spheres inside an external electric field

1. (1) Adopting the polar coordinates, the z component of the electric field produced by a dipole located at the origin with its axis parallel to the z axis is:

$$E_z(\theta,\phi) = -\frac{p}{4\pi\varepsilon_0}\frac{1-3\cos^2\theta}{r^3} \tag{2B.1}$$

where r is the length of the relative position vector of the two dipoles. In the external electric field ***E***, the energy of a dipole with its axis parallel to the z axis is:

$$U = -\vec{p}\cdot\vec{E} = -pE_z$$

Therefore we obtain that the interaction energy between two contacting small dielectric spheres is

$$U_{12} = \frac{p^2}{4\pi\varepsilon_0}\frac{1-3\cos^2\theta}{(2a)^3} \tag{2B.2}$$

(2) Based on Eq. (2B.2) for the configuration (a) we obtain:

$$U_a = \frac{1}{4\pi\varepsilon_0}\frac{1-3}{(2a)^3}p^2 = -\frac{1}{4\pi\varepsilon_0}\frac{p^2}{4a^3} \tag{2B.3}$$

For configuration (b)

$$U_b = \frac{1}{4\pi\varepsilon_0}\frac{1-0}{(2a)^3}p^2 = \frac{1}{4\pi\varepsilon_0}\frac{p^2}{8a^3} \tag{1B.4}$$

For configuration (c),

$$U_c = \frac{1}{4\pi\varepsilon_0}\frac{1-3\cos^2\pi/4}{(2a)^3}p^2 = -\frac{1}{4\pi\varepsilon_0}\frac{p^2}{16a^3} \tag{1B.5}$$

(3) Comparison between (2B.3)，(2B.4) and (2B.5) shows that configuration (a) has the lowest energy, corresponding to the ground state of the system.

2. With the similar approach to question 1 the interaction energies for the three different configurations can also be calculated.
For configuration (d),

$$U_d = \frac{1}{4\pi\varepsilon_0}\left(-\frac{p^2}{4a^3}+2\times\frac{p^2}{32a^3}\right) = -\frac{1}{4\pi\varepsilon_0}\frac{3p^2}{16a^3} \tag{2B.6}$$

For configuration (e),

$$U_e = \frac{1}{4\pi\varepsilon_0}\left(-\frac{p^2}{4a^3}\times 2-\frac{p^2}{32a^3}\right) = -\frac{1}{4\pi\varepsilon_0}\frac{17p^2}{32a^3} \tag{2B.7}$$

For configuration (f),

$$U_f = \frac{1}{4\pi\varepsilon_0}\left(\frac{p^2}{8a^3}\times 2 + \frac{p^2}{64a^3}\right) = \frac{1}{4\pi\varepsilon_0}\frac{17p^2}{64a^3} \tag{2B.8}$$

Comparison shows that configuration (e) of the lowest energy is most stable, while configuration (f) of the highest energy is most unstable.

Solution of the theoretical problem 3

3A. Average specific heat of each free electron at constant volume

(1). Each free electron has 3 degrees of freedom. According to the equipartition of energy theorem, at temperature T its average energy $\overline{E}$ equals to $\frac{3}{2}k_B T$, therefore the average specific heat c_v equals to

$$c_V = \frac{\mathrm{d}\overline{E}}{\mathrm{d}T} = \frac{3}{2}k_B$$

(2). Let U be the total energy of the electron gas, then

$$U = \int_0^S Ef(E)\mathrm{d}S$$

where S is the total number of the electronic states, E the electron energy.

Substitution of (1) for $\mathrm{d}S$ in the above expression gives

$$U = CV\int_0^\infty E^{3/2} f(E)dE = CVI\,,$$

where I represents the integral

$$I = \int_0^\infty E^{3/2} f(E)dE\,.$$

Usually at room temperature $k_B T << E_F$. Therefore, with the simplified $f(E)$

$$f(E) = \begin{cases} 1 & E < E_F - k_B T \\ -\dfrac{E-(E_F+k_B T)}{2k_B T} & E_F - k_B T < E < E_F + k_B T \\ 0 & E > E_F + k_B T \end{cases}$$

I can be simplified as

$$I = \frac{2}{5}E_F^{\ 3/2}(1-k_B T/E_F)^{5/2} + \frac{E_F + k_B T}{5k_B T}E_F^{\ 5/2}\left[(1+k_B T/E_F)^{5/2} - (1-k_B T/E_F)^{5/2}\right]$$

$$-\frac{1}{7k_B T}E_F^{\ 7/2}\left[(1+k_B T/E_F)^{7/2} - (1-k_B T/E_F)^{7/2}\right]$$

$$\approx E_f^{\ 5/2}\left[\frac{2}{5} + \frac{3}{4}\left(k_B T/E_F\right)^2\right]$$

Therefore

$$U = CVE_F^{\ 5/2}\left[\frac{2}{5}+\frac{3}{4}\left(k_B T / E_F\right)^2\right]$$

However the total electron number

$$N = CV\int_0^{E_F^0} E^{1/2}\mathrm{d}E = \frac{2}{3}CVE_F^{0\,3/2}$$

$$CV = \frac{3}{2}N\left(E_F^0\right)^{-3/2}$$

where E_F^0 is the Fermi level at 0K, leading to

$$U = \frac{3}{2}N\left(E_F^0\right)^{-3/2}E_F^{\ 5/2}\left[\frac{2}{5}+\frac{3}{4}\left(k_B T / E_F\right)^2\right]$$

Taking $E_F \approx E_F^0$, and $U = N\overline{E}$,

$$\overline{E} \approx \frac{3}{2}E_F\left[\frac{2}{5}+\frac{3}{4}\left(k_B T / E_F\right)^2\right]$$

$$c_v = \frac{\partial \overline{E}}{\partial T} = \frac{9}{4}k_B\frac{k_B T}{E_F} << \frac{3}{2}k_B$$

(3). Because at room temperature $k_B T$ =0.026eV while Fermi level of metals at room temperature is generally of several eVs, it can be seen from the above expression that according to the quantum theory the calculated average specific heat of each free electron at constant volume is two orders of magnitude lower than that of the classical theory. The reason is that with the temperature increase the energy of those electrons whose energy is far below Fermi level (several times of $k_B T$ less than E_F) does not change obviously, only those minor electrons of energy near E_F contribute to the specific heat, resulting in a much less value of the average specific heat.

Solution of the theoretical problem 3

3B. The Inverse Compton Scattering

1. Let p and E denote the momentum and energy of the incident electron, p' and E' the momentum and energy of the scattered electron, and $h\nu$ and $h\nu'$ the energies of the incident and scattered photon respectively. For this scattering process (see Fig. 1) energy conservation reads

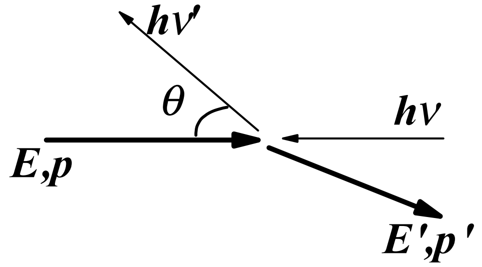


Figure 1

$$h\nu + E = h\nu' + E' \qquad (1B.1)$$

while the momentum conservation can be shown as (see Fig.2)

$$(p'c)^2 = (h\nu')^2 + (pc - h\nu)^2 + 2h\nu'(pc - h\nu)\cos\theta \qquad (1B.2)$$

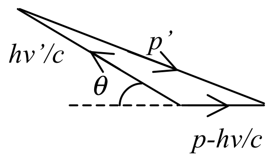


Figure 2

Equations (1B.1) and (1B.2), combined with the energy-momentum relations

$$E^2 = (pc)^2 + E_0^2 \qquad (1B.3)$$

and

$$E'^2 = (p'c)^2 + E_0^2 \qquad (1B.4)$$

lead to

$$h\nu' = \frac{E + pc}{E + h\nu + (pc - h\nu)\cos\theta} h\nu = \frac{E + \sqrt{E^2 - E_0^2}}{E + h\nu + (\sqrt{E^2 - E_0^2} - h\nu)\cos\theta} h\nu. \qquad (1B.5)$$

We have assumed that the kinetic energy of the incident electron is higher than its static energy, and the energy of the incident photon $h\nu$ is less than E_0, so that $\sqrt{E^2 - E_0^2} > h\nu$. Therefore from Eq. (1B.5), it can be easily seen that $\theta = \pi$ results in the maximum of $h\nu'$, and the maximum $h\nu'$ is

$$(h\nu')_{\max} = \frac{E + \sqrt{E^2 - E_0^2}}{E + 2h\nu - \sqrt{E^2 - E_0^2}} h\nu. \qquad (1B.6)$$

2. Substitution of $E=\gamma E_0$ into Eq. (1B.6) yields

$$(h\nu')_{\max}=\frac{\gamma E_0+\sqrt{\gamma^2-1}E_0}{\gamma E_0-\sqrt{\gamma^2-1}E_0+2h\nu}h\nu=\frac{\gamma+\sqrt{\gamma^2-1}}{\gamma-\sqrt{\gamma^2-1}+2h\nu/E_0}h\nu\ . \qquad (1B.7)$$

Due to $\gamma>>1$, $\sqrt{\gamma^2-1}\approx\gamma(1-\frac{1}{2\gamma^2})=\gamma-\frac{1}{2\gamma}$, and $h\nu/E_0<<1/\gamma$, then we have

$$(h\nu')_{\max}\approx\frac{\gamma+\gamma-1/2\gamma}{\gamma-\gamma+1/2\gamma+2h\nu/E_0}h\nu\approx4\gamma^2h\nu. \qquad (1B.8)$$

In the case of $\gamma=200$ and the wavelength of the incident photon $\lambda=500\text{nm}$

$$h\nu=\frac{hc}{\lambda}=\frac{1.24\times10^3}{500}=2.48\text{ eV},$$

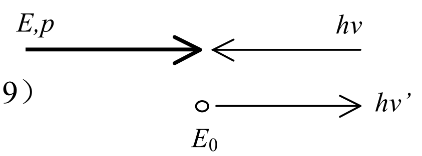
$$\frac{h\nu}{E_0}=\frac{2.48}{0.511\times10^6}=4.85\times10^{-6}<<\frac{1}{\gamma}=\frac{1}{200}=5.0\times10^{-3},$$

satisfying expression (1B.8). Therefore the maximum energy of the scattered photon

$$(h\nu')_{\max}\approx4\times200^2h\nu=1.6\times10^5\times2.48=3.97\times10^5\text{eV}\approx4.0\times10^5\text{eV}=0.40\text{MeV}$$

corresponding to a wavelength $\lambda'=\frac{hc}{h\nu'}=\frac{1.24\times10^3}{4.0\times10^5}=3.1\times10^{-3}\text{ nm}.$

3. (1) It is obvious that if the incident electron gives its total kinetic energy to the photon, the photon gains the maximum energy from the incident electron through the scattering process, namely the electron should become at rest after the collision. In this case, we have (see Fig. 3)



$h\nu+E=h\nu'+E_0$ (Conservation of energy) (1B.9)

$p-h\nu/c=h\nu'/c$ (Conservation of momentum)

or $pc-h\nu=h\nu'$ (1B.10)

Figure 3

Subtracting (1B.10) from (1B.9) leads to the energy of the incident photon

$$h\nu=\frac{1}{2}(E_0-E+pc)=\frac{1}{2}(E_0-E+\sqrt{E^2-E_0^2}). \qquad (1B.11)$$

In above equation the energy- momentum relation

$$(pc)^2 = E^2 - E_0^{\ 2} \qquad (1B.12)$$

has been taken into account. Therefore from Eq. (1B.9) we obtain the energy of the scattered photon

$$hv' = hv + E - E_0 = \frac{1}{2}(E - E_0 + \sqrt{E^2 - E_0^2}). \qquad (1B.13)$$

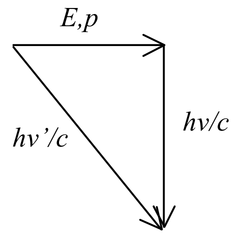


Figure 4

(2) Similar to question 3. (1), now we have (see Fig. 4)

$$h\nu + E = h\nu' + E_0 \qquad \text{(Conservation of energy)} \quad (1B.9)$$

$$p^2 + (h\nu / c)^2 = (h\nu' / c)^2 \qquad \text{(Conservation of momentum)}$$

That is, $(pc)^2 + (h\nu)^2 = (h\nu')^2$. (1B.14)

Substitution of Eq. (1B.12) into Eq. (1B.14) yields

$$E^2 - E_0^{\ 2} + (h\nu)^2 = (h\nu')^2 .$$

On the other hand, square of Eq. (1B.9) results in

$$(h\nu')^2 = E^2 + E_0^{\ 2} + (h\nu)^2 + 2Eh\nu - 2EE_0 - 2E_0 h\nu$$

Combination of the above two equations leads to

$$E^2 + E_0^{\ 2} + (h\nu)^2 + 2Eh\nu - 2EE_0 - 2E_0 h\nu = E^2 - E_0^{\ 2} + (h\nu)^2$$

That is, $2(E - E_0)h\nu = 2E_0(E - E_0)$.

Then, we obtain the energy of the incident photon

$$h\nu = E_0 . \qquad (1B.15)$$

Substitution of (1B.15) into Eq. (1B.9) gives the energy of the scattered photon

$$h\nu' = h\nu + E - E_0 = E \qquad (1B.16)$$

Explanatory notes about the solution of Question 3:

Question 3. (1) can also be solved as follows. According to Eq. (1B.6), the maximum energy Δ that the photon of energy $h\nu$ gains from the electron is

$$\Delta = hv'_{\max} - hv = 2\frac{pchv - h^2v^2}{E + 2hv - pc},$$

where $pc=\sqrt{E^2-E_0^2}$. To obtain the maximum Δ, we use the extreme condition

$$\frac{d(\Delta/2)}{d(hv)}=\frac{(pc-2hv)(E+2hv-pc)-2(pchv-h^2v^2)}{(E+2hv-pc)^2}=0\,.$$

Let the numerator equal to zero, a quadratic equation results:

$$2(hv)^2+2(E-pc)hv-(Epc-p^2c^2)=0\,.$$

Its two roots can be shown as

$$hv=\frac{1}{2}\left[-(E-pc)\pm\sqrt{E^2-p^2c^2}\right]=\frac{1}{2}(-E+pc\pm E_0)\,.$$

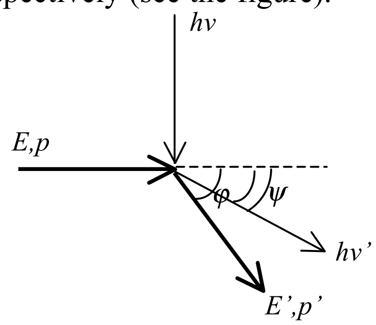
Since the negative sign leads to a meaningless negative hv, we have

$$hv=\frac{1}{2}(\sqrt{E^2-E_0^2}-E+E_0)\,.$$

where $pc=\sqrt{E^2-E_0^2}$ has been taken into account. This result is just the same as Eq. (1B.11). The expression for hv' is then the same as Eq. (1B.13).

<u>Question 3. (2) can also be solved as follows</u>

For the sake of simplification, it is assumed that the scattered photon and electron move in the same plane which the incident photon and electron moved in. Meanwhile the angles which the directions of the scattered photon and electron make with the direction of the incident electron are denoted by ψ and φ respectively (see the figure).



Then, we have

$E+hv=E'+hv'$, (Conservation of energy) (1B.1')

$$p=\frac{hv'}{c}\cos\psi+p'\cos\varphi,$$

or

$pc-hv'\cos\psi=p'c\cos\varphi,$ (Conservation of horizontal momentum) (1B.2')

and $\frac{hv}{c}=\frac{hv'}{c}\sin\psi+p'\sin\varphi,$

or

$hv-hv'\sin\psi=p'c\sin\varphi\,.$ (Conservation of vertical momentum) (1B.3')

$(1B.2')^2$+$(1B.3')^2$ leads to

$$p'^2c^2 = p^2c^2 + (hv')^2\cos^2\psi - 2pchv'\cos\psi + (hv)^2 + (hv')^2\sin^2\psi - 2hvhv'\sin\psi$$

(1B.4')

Square of Eq.(1B.1') results in

$$E'^2 = E^2 + (hv)^2 + (hv')^2 + 2Ehv - 2Ehv' - 2hvhv'.$$

Substitution of the energy-momentum relation $E'^2 = E_0^2 + p'^2c^2, E^2 = E_0^2 + p^2c^2$ into the above equation of energy conservation leads to

$$p'^2c^2 = p^2c^2 + (hv)^2 + (hv')^2 + 2Ehv - 2Ehv' - 2hvhv' \qquad (1B.5')$$

Comparison between Eq.（1B.4'）and Eq. (1B.5') yields the energy of the scattered photon

$$hv' = \frac{Ehv}{E + hv - (pc\cos\psi + hv\sin\psi)}. \qquad (1B.6')$$

From (1B.6') it can be seen that if $\psi = \cos^{-1}\frac{pc}{\sqrt{p^2c^2 + h^2v^2}}$, the energy of the scattered photon reaches the maximum,

$$hv'_{\max} = \frac{Ehv}{E + hv - \sqrt{p^2c^2 + h^2v^2}}. \qquad (1B.7')$$

The energy that the photon gets from the electron is

$$\Delta = hv'_{\max} - hv = \frac{\sqrt{p^2c^2 + h^2v^2}\,hv - (hv)^2}{E + hv - \sqrt{p^2c^2 + h^2v^2}}. \qquad (1B.8')$$

The extreme condition for Δ is

$$\frac{d\Delta}{d(hv)} = \frac{A}{(E + hv - a)^2} = 0, \qquad (1B.9')$$

where $\sqrt{p^2c^2 + h^2v^2} = a$

and $A = (a + \frac{(hv)^2}{a} - 2hv)(E + hv - a) - (hva - h^2v^2)(1 - \frac{hv}{a})$.

$A = 0$ results in

$$Ea + hva - p^2c^2 - (hv)^2 + \frac{E(hv)^2}{a} + \frac{(hv)^3}{a} - (hv)^2 - 2hvE - 2(hv)^2 + 2hva$$
$$= hva - (hv)^2 - (hv)^2 + \frac{(hv)^3}{a}$$

Simplifying this equation leads to

$$Ea - p^2c^2 + \frac{E(hv)^2}{a} - 2hvE - 2(hv)^2 + 2hva = 0$$

i.e., $$(E + 2hv)a + \frac{E(hv)^2}{a} = p^2c^2 + 2hvE + 2(hv)^2 .$$

Squaring both of the two sides of this equation yields

$$(E + 2hv)^2 a^2 + \frac{E^2(hv)^4}{a^2} + 2(E + 2hv)E(hv)^2$$
$$= (pc)^4 + 4(hv)^2 E^2 + 4(hv)^4 + 4(pc)^2 hvE + 4(pc)^2(hv)^2 + 8(hv)^3 E$$

Substitution of $\sqrt{p^2c^2 + h^2v^2} = a$ into the above equation and making some simplifications yield

$$E^2p^2c^2 + \frac{E^2(hv)^4}{p^2c^2 + h^2v^2} = p^4c^4 + h^2v^2E^2 ,$$

that is,

$$E^2(pc)^4 + E^2(pc)^2(hv)^2 + E^2(hv)^4 = (pc)^6 + (pc)^4(hv)^2 + E^2(pc)^2(hv)^2 + (hv)^4E^2 .$$

After some simplifications we obtain

$$(pc)^4(hv)^2 = (pc)^4(E^2 - p^2c^2) = (pc)^4 E_0^2 ,$$

which yields $$hv = E_0. \qquad \text{(1B.10')}$$

Substitution of (1B.10') into Eq. (1B.7') leads to

$$hv'_{\max} = \frac{EE_0}{E + E_0 - E} = E . \qquad \text{(1B. 11')}$$

The results (1B.10') and (1B.11') are just the same as Eqs. (1B.15) and (1B.16) in the former solution.