
Theoretical Problem 1. Tea Ceremony and Physics of Bubbles

The tea ceremony is traditional in Asia. One of the important steps in preparation of tea is the boiling of fresh water when bubbles appear inside. Bubbles are familiar from daily life and occupy an important role in physics, chemistry, medicine and technology. Nevertheless, their behavior is often surprising and unexpected - and, in many cases, still not understood.

At room temperature the pure water is saturated with gas. With increasing temperature the excess pressure of dissolved gas P_{ab} increases, the dissolved air is liberated and air bubbles (**ABs**) appear at the bottom and walls of teakettle (Fig.2). For pure water the wettability is sufficient and an AB represents a truncated sphere with radius R_{ab} and with unwetted foundation with radius $r_{ab} << R_{ab}$. At more heating ABs expand and by reaching certain sizes can detach from the bottom (Fig.3), flow up to the water surface and burst there. The vapor bubbles (**VBs**) appear when the water temperature at the bottom reaches the critical value $T_w^0 \approx T_{crit}^0 = 100^0 C$ at which the pressure of the saturated vapor exceeds the external pressure. The vapor production increases tens times, VBs expand and detach from the bottom. VB may be considered consisting only of vapor. If the water is heated sufficiently, the uprising VB continue to swell, reach the surface and burst. Else, water is not heated enough in the higher layers and there exits a vertical strong temperature gradient. By reaching relatively cold layers of water VB collapse in the volume of water (Fig.4). This causes the induced degassing - strong oscillations and a considerable amount of dissolved air is released in the form of microscopic air bubbles (**MAB**). This can generate ultrasonic vibrations.

The main stages of the bubble evolution during the boiling process are:
- the appearance and growth of AB at the bottom and walls, their transmutation into VB;
- the detachment and uprising of VB, their disappearance in the water volume or at the surface;
- the appearance of MAB in the water volume and their uprising to the surface.

This theoretical description is in good agreement with modern experiments. Particularly, an interesting noise analysis experiment (**NAE,** Ural State University, Ekaterinburg) for the boiling water was performed. Highly sensitive microphones attached to wide-band amplifiers and brought to an electric teakettle have detected three main origins of noises:

1. AB's detachments from the bottom before boiling (generate oscillations with $\nu_1 \sim 100$ Hz,);
2. VB's collapses in the volume of water (generate oscillations with $\nu_2 \sim 1$ kHz);
3. MAB's appearances under the water surface (generate oscillations with ν_3~35kHz to 60kHz).

Hints:

1) It is well known that a small bubble rises along a rectilinear path and a laminar flow is observed - water flows easy and layer-wise (see Fig.1). Then, the Stokes formula describes the dissipative force for a particle moving with slow velocity v_{lam} *:*

$$F_A = 6\pi\eta_w R_b v_{lam}$$

In contrast to this picture, when relatively large bubbles lift to the surface, it disturbs the surrounding water, cavitation hollows appear behind and the turbulent flow is observed (see Fig.1). In this case a part of the kinetic energy of an uprising bubble transfers into the dissipative work.

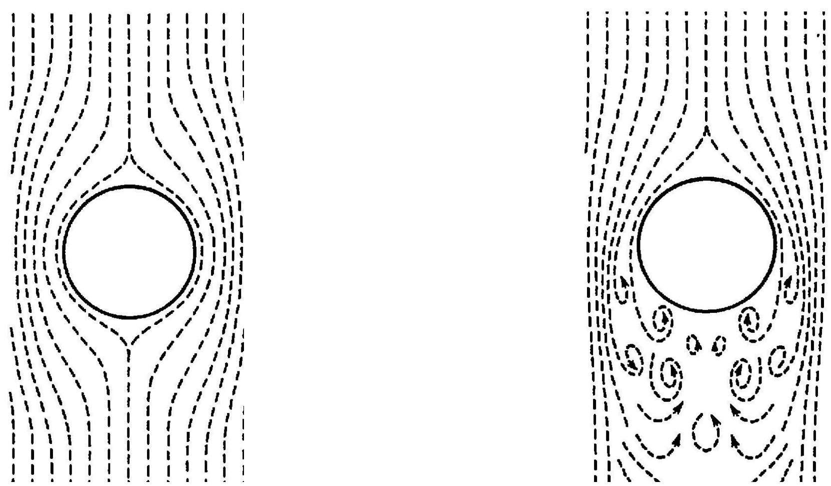

Fig 1. Laminar and turbulent types of flow for rising air bubbles in water

2) When the surface of liquid has a convex (concave) form there appears a surface tension force due to molecular interaction near the edge. This pressure can be given by formula

$$\Delta P = \frac{2\sigma}{R}$$

where σ *- is the surface tension coefficient (unit=N/m), the force coming to unit length of surface, R – is the radius of surface curvity.*

3) When dealing with a short process with characteristic duration time "t", its inverse value may be considered as a characteristic frequency $\nu = \frac{1}{t}$. *Use this definition for calculating the noise frequencies.*

Useful data:

$P_0 = 1.016 \cdot 10^5 [Pa]$ - atmospheric pressure,

$\rho_w = 10^3 [kg/m^3]$ - water density,

$\rho_{vapor} = 0.017 [kg/m^3]$ - vapor density at T=293K; ($= 0.596 [kg/m^3]$ at T=373K)

$P_{vapor} = 0.023 \cdot 10^5 [Pa]$- vapor pressure at T=293K; ($= 1.016 \cdot 10^5 [Pa]$ at T=373K)

$g = 9.81 [m/s^2]$ - acceleration of gravity,

$\mu_{air} = 0.029 [kg/mole]$ - molecular weight of air

$R = 8.31 [J/mole/K]$ - the gas universal constant

$\sigma = 0.0725 [N/m]$ - surface tension coefficient of water,

$\eta_w = 0.3 \cdot 10^{-3} [Pa \cdot s]$ - coefficient of viscosity of water

H=10cm – Water attitude in teakettle

Fig. 2. Bubbles in teakettle

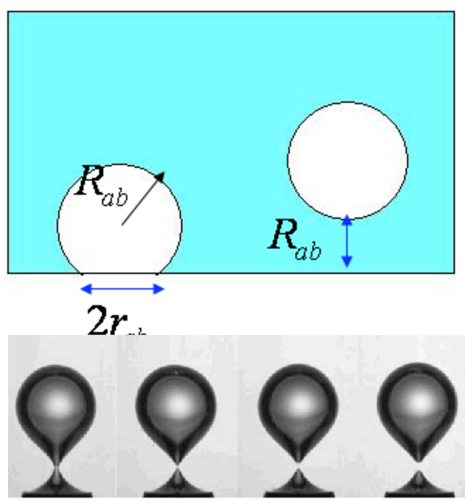

Fig. 3. Air bubble detaching from the bottom

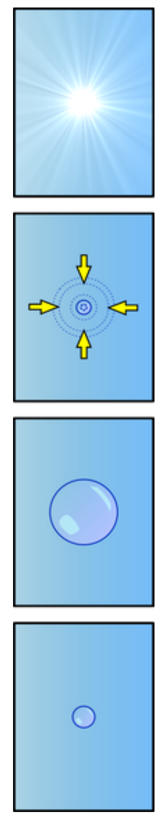

Fig. 4. Vapor bubble collapsing

Questions (total 10 points):

Consider water boiling in a flat-bottomed cylinder glass teakettle at normal atmospheric pressure. The bottom of the kettle heats up uniformly and a vertical temperature gradient exists, bubbles appear and evolute (Fig.2).

Q1. Write the pressure condition of the growth of an AB in the water volume at height **h<H**, where H is the water surface level in the teakettle. Take into account the inequality $2\pi\sigma r_{ab} >> P_{extern}\pi r_{ab}^2$. *[in terms of* $P_{ab}, P_0, R_{ab}, \rho_w, g, \sigma, h, H$ *]* (1.0 point)

Q2. Write for an AB the condition of the detachment from the bottom of the teakettle (Fig.3).
Take into account the relation $r_{ab} << R_{ab}$. *[in terms of* $r_{ab}, R_{ab}, \rho_w, \sigma$ *]* (1.5 points)

Q3. Consider an AB with radius R_b at the bottom of the teakettle. As water is boiled, the bubble is saturated with vapor and enlarges its radius. Write the ratio $\xi \equiv m_{air} / m_{vapor}$ of the masses of the air and saturated vapor inside the bubble at given temperature T. Calculate the ratio at room temperature T=20^{o}C ($R_b = 0.5mm$) and at boiling point at T=100^{o}C ($R_b = 1mm$). *[in terms of* $\mu_{air}, T, P_0, P_{vapor}, R_b, \rho_w, \rho_v, \sigma, H$ *]* (1.5 points)

Q4. By using the NAE data and Newton's Law estimate the radius of the AB detached from the bottom and uprised in distance R_{ab} (Fig.3). Assume, that the added-mass (taking into account surrounding water layer) of AB is a half of the analogous water bubble. (1.0 points)

Q5. Write the radius of the foundation of an AB just before the uprising, when the connecting "neck" is very narrow (see Fig.3). *[in terms of* R_{ab}, ρ_w, σ *]*. Calculate it by using the radius found in Q4. (1.5 points)

Q6. By using the NAE data estimate the radius of collapsing VB (Fig.4) by assuming that the radial pressure is about **3kPa** during this process. (1.2 points)

Q7. By using previous results for VB calculate the radius of the MAB produced during induced degassing. (0.5 points)

Q8. Write the uprising velocity for typical AB by using the Stokes law of a laminar flow.
[in terms of R_{ab}, ρ_w, η_w*]*. Estimate the uprising time for H=10cm. (0.6 points)

Q9. Write the average velocity of the elevation of VB with turbulent type of flow
[in terms of R_{ab}, ρ_w, η_w*]*. Estimate the uprising time for H=10cm. (1.2 points)

Problem 2: Ionic Crystal, Yukawa-type Potential and Pauli Principle

Atoms of many chemical elements possess very low ionization energy and easily loose the outer electrons. Vice versa, atoms of other elements accept easily the electrons. Taken into one volume, these positive and negative ions can combine into stable ionic structures. Many solids exhibit a crystal structure, in which the atoms are arranged in extremely regular, periodic patterns. In an ideal crystal the same basic structural unit is repeated through the space.

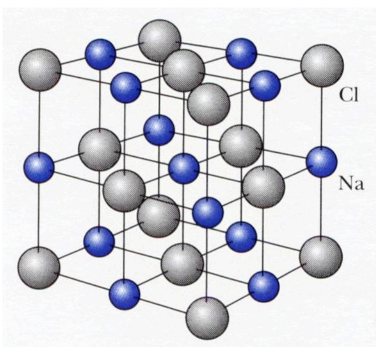

Fig.1

The face-centered cubic lattice of the sodium chloride (NaCl). The lattice spacing between the atomar centres is constant and is given by parameter r_0.

The main contribution to the binding energy of an ionic crystal is given by the electrostatic potential energy of ions.

The electric interaction acting between two point charges q_1 and q_2 standing in distance R is well defined by Coulomb's potential:

$$V_C(R) = k\frac{q_1 q_2}{R}$$

where $k = 1/4\pi\varepsilon_0 \approx 9\cdot 10^9 [N\cdot m^2/C^2]$ is the Coulomb constant. A negative force implies an attractive force. The force is directed along the line joining the two charges. For the case of NaCl crystals both types of ions has the unit charge **±e** and one should also take into account many other neighbors acting on the chosen ion. Taking into account all positive and negative ions in a crystal of the infinite sizes results in the **attractive** potential energy $V_{att}(r) = \alpha\cdot V_C(r)$, where r is the distance between nearest neighbors and α=1.74756 is the Madelung constant [*E.Madelung, Phys. Zs, 19 (1918) p542*] and used in determining the energy of a single ion in a crystal.

Along the attractive potential energy there should to be a **repulsive** potential energy due to the Pauli Exclusion Principle and the overlap of electron shells in a crystal lattice. In contrast to the Coulomb-type attractive part, the repulsive potential energy is very short range.

There are two different models to describe the repulsive potential.

Model #1. A reasonable approximation to the repulsive potential is an exponential function:

$$V_{rep1}(r) = \lambda \cdot e^{-r/\rho}, \qquad (\lambda, \rho > 0)$$

which describes the repulsion interaction of selected ion with the entire crystal lattice. Here λ is coupling strength and ρ stands for the range parameter.

Model #2. Another good approximation to the repulsive potential is an inverse power

$$V_{rep2}(r) = b / r^n, \qquad (b > 0)$$

where b is coupling strength and n is integer greater than 2 (the Born exponent). These parameters take into account the repulsion with entire crystals.

Obviously, the physical parameters and model potentials depend on the type of crystal lattice.

Experimental data for the lattice constant r_0 and the dissociation energy E_{dis} (needed to break the lattice into separate ions) are given in Table 1 for some ionic crystals at normal temperature and pressure.

Crystal	r_0 [nm]	E_{dis} $[kJ/mole]$
NaCl	0.282	+764.4
LiF	0.214	+1014.0
RbBr	0.345	+638.8

Table 1
Properties of Salt Crystals with the NaCl Structure
[C.Kittel, "Introduction to Solid State Physics", N.Y., Wiley (1976) p.92] (in one mole there are the Avogadro's number of pairs of ions or atoms).

Questions (total 10 points):

Q1. Write down the Coulomb potential $V_{C0}(r)$ for an ion located at the centre of cubic lattice in Fig 1. Let it interact only with the nearest neighbors (in distance up to and including $r=\sqrt{3}r_0$) of a crystal lattice. Find the Madelung constant α_0 corresponding to this approximation.
(1.5 points)

Q2. By using Model #1 write down the net potential energy per ion $V_1(r)$. Determine its equilibrium equation for $r=r_0$ and write down the net potential energy $V_1(r_0)$
[in terms of α, r_0, ρ *]***.** Use exact Madelung constant α. (1.5 points)

Q3. By using experimental data estimate the range parameter ρ. Use $N_A = 6.022\cdot10^{23}[1/mole]$.
(2.0 points)

Q4. By using Model #2 write down the net potential energy $V_2(r)$ per ion. Determine its equilibrium position $r=r_0$ and write down the net potential energy $V_2(r_0)$**.** Use exact Madelung constant α. *[in terms of* α, r, r_0, n *]* (2.0 points)

Q5. By using experimental data (from Table 1) estimate the Born exponent n for NaCl**.** Estimate the proportions of the Coulomb interaction and the Pauli exclusion (repulsive part) in the entire net potential energy in the equilibrium state? (1.5 points)

Q6. The ionization energy (required to extract an electron from an atom) of the Na atom is +**5.14 eV**, the electron affinity (required to receive an electron to an atom) of the Cl atom is **-3.61 eV**. Estimate the total binding energy (holds an atom inside lattice) per atom in the NaCl crystal. The experimental result is $E_{\exp}\approx -3.28[\mathrm{eV}]$. *[in terms of eV]*

Use the conversion of units: $1[\mathrm{eV}] = 1.602\cdot10^{-19}[\mathrm{J}]$ (1.5 points)

Problem 3. How does a superluminal object look like?

Can a body move faster than the speed of light? The answer is “No” if the object is moving in the vacuum. But the answer can be “Yes”, if we deal with the phase speed of light in an optically dense medium with refractive index of n ($n = c/u$, where u is the speed of light in the medium, and c is the speed of light in the vacuum).

We say a body is superluminal, if $u < v < c$, where v is the velocity of the body. One of the well known examples of the superluminal body is a charged particle generating Cherenkov radiation.

Throughout the problem we will deal with a superluminal body of constant velocity v in an optical medium without dispersion. u is the velocity of light in the medium.

For the simplicity, we introduce a notation $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$ and an angle θ given by $\cos\theta = u/v$

and $\tan\theta = \sqrt{\frac{v^2}{u^2} - 1}$.

1. Radiating superluminal particle

As shown in Fig.1, a radiating particle is moving along the x-axis with a constant velocity v ($v > u$).

An observer M is located at the distance d from x-axis.

We choose the point nearest to the observer as the point O, the origin on the x-axis. The time when the particle actually passes over the point x=0 is taken to be t=0.

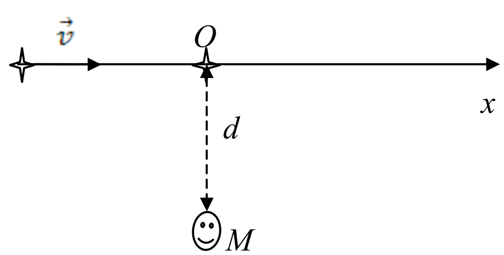

Figure 1

(1)Suppose the light radiated at the given time t' is observed at time t. Express t in terms of d, t', u and v. (1.0 point)

(2)At time $t = t_0$, the observer first sees the particle at position x'_0. Find the apparent position x'_0 and the observed time t_0 for this first appearance in terms of d, v and θ. (2.0 points)

(3) Find the apparent position(s) x' of the particle for any given time t. Write your answer in terms of v, θ, t and t_0. (2.0 points)

(4) Find the apparent velocity(s) $v'(t)$ of the particle for any given time t. Write your answer in terms of v, θ, t, and t_0. (1.0 point)

(5) Find the apparent velocity(s) v' of the first appearance of the particle. (0.2 points)

(6) Find the apparent velocity(s) v' of the particle at infinite distances from the origin, *O*. Write your answer in terms of v and u. (0.2 points)

(7) Sketch the graph of the apparent velocity v' versus time *t*, indicating clearly asymptotic values of the apparent velocity. (1.0 point)

(8) Can an apparent velocity exceed the light speed in the vacuum, i.e. $v' > c$? (0.2 points)

2.Radiating linear object

Consider a linear object, radiating light and moving along the *x*-axis. The length of the linear object is L in the rest frame of the object.

A. Parallel movement

In this section, we assume that the radiating linear object moves longitudinally along *x*-axis as shown in Fig.2.

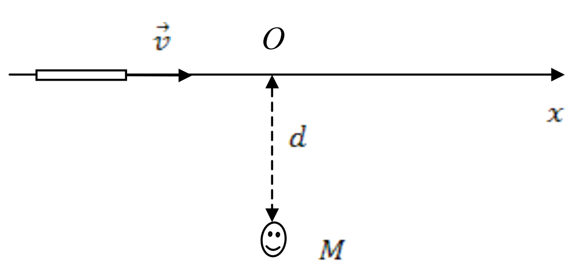

Figure 2

(9) Determine the time interval of complete appearance of the whole linear object from the first appearance of its front point. Write your answer in terms of L, γ and v. (0.3 points)

(10) Determine the apparent length(s) of the object at the moment of its complete appearance. Write your answer in terms of d, L, θ and γ. (0.4 points)

B. Perpendicular movement

In this section, we assume that the radiating linear object moves perpendicularly along x-axis as shown in Fig.3. Let the observer be located at the origin of x-axis $(d = 0)$. The object is symmetrical with respect to x-axis.

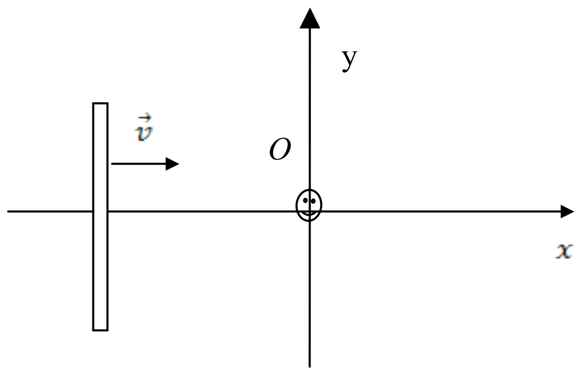

Figure 3

(11) Show that for a given time t, the apparent form of this object is an ellipse or part(s) of an ellipse. (0.7 points)

Find the following quantities and express them in terms of v, θ, and t.

(12) Find the position x_c of the centre of symmetry of the ellipse for a given time t in terms of v, θ and t. (0.5 points)

(13) Determine the lengths of the semi-major and semi-minor axes of the ellipse for a given time t in terms of v, θ and t. (0.5 points)