Solutions:

S1. The condition of the survival and growth for AB appeared in the water volume at height $h < H$ is the competiveness of the pressures acting inside and outside (atmospheric, hydrostatic and surface tension) on the bubble surface:

$$P_{ab} = P_{in} \geq P_{out} \equiv P_0 + g\rho_w \cdot (H-h) + \frac{2\sigma}{R_{ab}},$$

S2. The Archimedes lifting force is

$$F_A = g \cdot (\rho_w - \rho_{ab}) \frac{4\pi R_{ab}^3}{3} \approx g \cdot \rho_w \frac{4\pi R_{ab}^3}{3}$$

where $\rho_{ab} \ll \rho_w$ - the air density in bubble.

For $r_{ab} \ll R_{ab}$ the Laplace surface tension force plays main role in holding down the bubble:

$$F_{down} = \sigma \cdot 2\pi r_{ab}$$

The stability of the AB at the bottom means:

$$F_A = F_{down}$$

At more heating the lifting force overbalances the holding force:

$$g\frac{4\pi R_{ab}^3}{3}\rho_w > \sigma \cdot 2\pi r_{ab}$$

and AB is detached from the bottom and floats (Fig.3).

S3. The air and vapor pressures inside compensate the outer pressure for a stable bubble.

$$P_{vapor} + P_{air} = P_0 + g\rho_w H + \frac{2\sigma}{R_b}$$

The mass of air inside AB may be found from the Clayperon-Mendeleev equation

$$m_{air} = \frac{\mu_{air} P_{air} V}{RT}$$

The vapor mass is

$$m_{vapor} = \rho_{vapor} \cdot V$$

Then the ratio is

$$\xi \equiv \frac{m_{air}}{m_{vapor}} = \frac{\mu_{air} P_{air}}{RT\rho_v} = \frac{\mu_{air}}{RT\rho_v}\left(P_0 + g\rho_w H + \frac{2\sigma}{R_b} - P_{vapor}\right)$$

For $T = 20^0 C = 293K$ and $R_b = 0.5mm$:

$$\xi = \frac{0.029}{8.31 \cdot 293 \cdot 0.0173}\left(1.016 \cdot 10^5 + 9.81 \cdot 10^3 \cdot 0.1 + \frac{2 \cdot 0.0727}{0.5 \cdot 10^{-3}} - 2.3 \cdot 10^3\right) \approx 69.2$$

This bubble consists mostly of air.

For $T = 100^0 C = 373K$ and $R_b = 1mm$:

$$\xi = \frac{0.029}{8.31 \cdot 373 \cdot 0.596}\left(1.016 \cdot 10^5 + 9.81 \cdot 10^3 \cdot 0.1 + \frac{2 \cdot 0.0588}{1 \cdot 10^{-3}} - 1.016 \cdot 10^5\right) \approx 0.017$$

This bubble consists mostly of saturated vapor.

S4. Consider AB detached from the bottom and uprising in distance R_{ab} (see Fig.3). During the rise, bubbles induce a displacement of the surrounding fluid in their vicinity, which leads to an added-mass force. The added-mass of a bubble is

$$M_{invol} \approx \frac{V_{ab}\rho_w}{2}$$

The acceleration at the detachment moment is

$$a_{\det} = \frac{F_A}{M_{invol}} = \frac{gV_{ab}\rho_w}{V_{ab}\rho_w / 2} = 2g$$

The characteristic "detachment time"

$$t_1 = \sqrt{\frac{2R_{ab}}{a_{\det}}}$$

defines the duration of the impact to the liquid during the detachment. The liquid begins to vibrate with characteristic frequency

$$\nu_1 = \frac{1}{t_1} = \sqrt{\frac{g}{R_{ab}}}$$

Substituting the date from NAE we estimate the characteristic radius of detaching AB

$$\nu_1 \approx 100[Hz] \quad \Rightarrow \quad R_{ab} \approx 1 \cdot 10^{-3}[m]$$

S5. The balance of the lifting and confining forces reads

$$gV_{ab}\rho_w \approx \sigma \cdot 2\pi r_{ab}$$

Find the typical foundation radius:

$$r_{ab} \approx \frac{2g\rho_w R_{ab}^3}{3\sigma}$$

For AB with radius ~***1mm*** we calculate

$$r_{ab} \approx \frac{2 \cdot 9.81 \cdot 10^3 \cdot 10^{-9}}{3 \cdot 0.0725}[m] \approx 9.02 \cdot 10^{-5}[m]$$

S6. Consider collapsing VB during time piece t_2 (Fig.4). Let's estimate the characteristic frequency of the ultrasonic shock waves. The surrounding water flood the collapsed VB with acceleration **a** and the Newton equation reads

$$F_{collap} \approx a \cdot \frac{4\pi R_{cb}^3}{3} \rho_w$$

where the acceleration of the water front converging in the center of VB is

$$a = \frac{2R_{cb}}{t_2^2} = 2R_{cb} \nu_2^2$$

The radial pressure on the VB surface is given

$$\Delta P = \frac{F_{collap}}{4\pi R_{cb}^2} = \frac{2R_{cb}^2}{3} \rho_w \nu_2^2$$

Particularly, for the obtained data of the NAE experiment we obtain

$$R_{cb} = \frac{1}{\nu_2} \sqrt{\frac{3\Delta P}{2\rho_w}}$$

Particularly,

$$\nu_2 \approx 1[kHz] \;,\; \Delta P = 3[kPa] \;\Rightarrow\; R_{cb} \approx 3[mm]$$

This result is in good agreement with another experiment (see Fig.5) where the radius is found
about **3.5mm** and the collapsing time is about **1ms** (i.e., **~1kHz** noise).

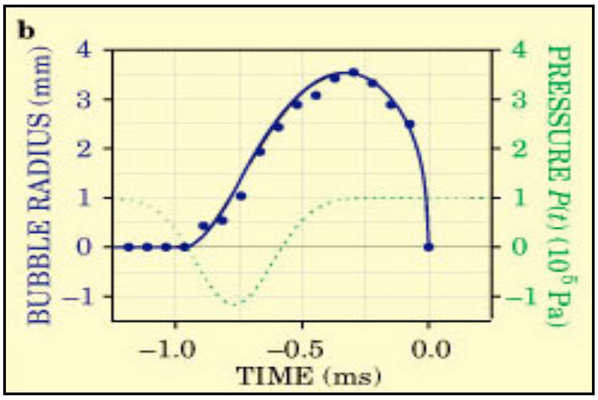

Fig.5. A vapor bubble collapsing evolution. [M. P. Brenner, S. Hilgenfeldt, D. Lohse, *Rev. Mod. Phys.* **74**, 425 (2002)]

S7. Obviously, the physical nature of MAB is the same as for VB. Then,

$$\frac{\nu_3}{\nu_2} = \frac{R_{cb}}{R_{mab}} \quad \Rightarrow \quad R_{mab} = R_{cb} \frac{1[kHz]}{35 \div 60[kHz]} \approx \frac{3\,[mm]}{35 \div 60} \approx 0.05 \div 0.086[mm]$$

S8. A bubble detached from the bottom is hoisted under the influence of Archimedes force. The
water resistant force depends on the nature of the streamline flow (Figs.1,6,7).

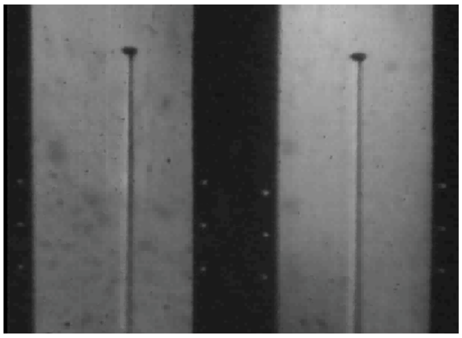

Fig. 6 The rectilinear air bubble trajectory (R= 0.69mm) rising from the bottom. On the left the XZ view and on the right the YZ view. The black areas are part of the reference system outside the water tank. [Benjamin(1987), A.de Vries (2001)]

But for a bubble with radius about **1mm** the emersion laminar velocity becomes too fast

$$v_{lam} \approx 2.2\,[m/s]$$

and the bubble passes the 10cm distance to the surface for 0.045 second !!!. It is obviously wrong. Therefore, the Stokes formula is not applicable for AB and VB rising from the bottom.

S9. As a path instability sets in, a bubble can uprise by either zigzag or spiral (Fig.8).

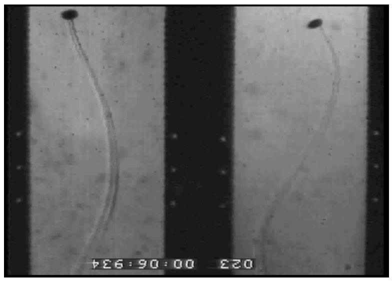

Fig.7 The trajectory of a spiralling bubble in two perpendicular views, XZ and YZ (R=1.1mm) [Benjamin(1987), Antoine de Vries (2001)].

Obviously, the lifting force is

$$F_A = \frac{4\pi R_{vb}^3}{3} g\rho_w$$

During its uprising in distance 'h' the bubble removes a water portion with mass:

$$M_{turb} = \rho_w h \cdot \pi R_{vb}^2$$

and performs a work (transfers kinetic energy)

$$W_{dissip} = \frac{M_{turb} v_{turb}^2}{2}$$

Then, we estimate the dissipative force

$$F_{dissip} = \frac{W_{dissip}}{h} = \frac{M_{turb} v_{turb}^2}{2h}$$

Since VB flows steady without any acceleration, the dissipative (braking) force balances the Archimedes force:

$$F_{dissip} = F_A$$

Then, the average (for spiral motion) turbulent velocity is

$$v_{turb} \approx \sqrt{\frac{8gR_{vb}}{3}}$$

For a typical radius of uprising AB **(~1mm**) or collapsing VB (~3mm) we calculate the velocity

$$v_{turb} \approx \sqrt{\frac{8 \cdot 10 \cdot (1 \div 3) \cdot 10^{-3}}{3}}[m/s] \approx (0.16 \div 0.28)[m/s]$$

The time required to pass a distance H**~10cm**:

$$t_{turb} > \frac{H}{v_{turb}} \approx \frac{0.1}{(0.16 \div 0.28)}[s] \approx (0.35 \div 0.62)[s]$$

This is a quite reasonable result and the bubbles mostly elevate under the turbulence flow law.

[*Marking Scheme*] Tea Ceremony and Physics of Bubbles

Q	Item	Answer	Points
1	condition of growth	$P_{ab} = P_{in} \geq P_{out} \equiv P_0 + g\rho_w \cdot (H-h) + \frac{2\sigma}{R_{ab}}$	1.0
2	condition of detachment	$g\frac{4\pi R_{ab}^3}{3}(\rho_w - \rho_{ab}) > \sigma \cdot 2\pi r_{ab} + (P_0 + g\rho_w H)\cdot \pi r_{ab}^2$	1.5
3	the ratio	$\xi \equiv \frac{m_{air}}{m_{vapor}} = \frac{\mu_{air}P_{air}}{RT\rho_v} = \frac{\mu_{air}}{RT\rho_v}\left(P_0 + g\rho_w H + \frac{2\sigma}{R_b} - P_{vapor}\right)$	1.1
	the ratio at T=20C	$\xi = \frac{0.029}{8.31\cdot 293\cdot 0.0173}\left(1.016\cdot 10^5 + 9.81\cdot 10^3\cdot 0.1 + \frac{2\cdot 0.0727}{0.5\cdot 10^{-3}} - 2.3\cdot 10^3\right) \approx 69.2$	0.2
	the ratio at T=100C	$\xi = \frac{0.029}{8.31\cdot 373\cdot 0.596}\left(1.016\cdot 10^5 + 9.81\cdot 10^3\cdot 0.1 + \frac{2\cdot 0.0588}{1\cdot 10^{-3}} - 1.016\cdot 10^5\right) \approx 0.017$	0.2
4	characteristic frequency	$\nu_1 = \frac{1}{t_1} = \sqrt{\frac{g}{R_{ab}}}$	0.8
	radius of detaching	$R_{ab} \approx 1\cdot 10^{-3}[m]$	0.2
5	foundation radius:	$r_{ab} \approx \frac{2g\rho_w R_{ab}^3}{3\sigma}$	1.2
	For radius 1mm	$r_{ab} \approx \frac{2\cdot 9.81\cdot 10^3\cdot 10^{-9}}{3\cdot 0.0725}[m] \approx 9.02\cdot 10^{-5}[m]$	0.3
6	Radius of collapsing bubble	$R_{cb} = \frac{1}{\nu_2}\sqrt{\frac{3\Delta P}{2\rho_w}}$	1.0
	Numerical value	$\nu_2 \approx 1[kHz]$, $\Delta P = 3[kPa] \Rightarrow R_{cb} \approx 3[mm]$	0.2
7	radius	$R_{mab} = R_{cb}\frac{1[kHz]}{35\div 60[kHz]} \approx \frac{3[mm]}{35\div 60} \approx 0.05\div 0.086[mm]$	0.5
8	laminar velocity	$v_{lam} \approx 2.2\,[m/s]$	0.6
9	turbulent velocity	$v_{turb} \approx \sqrt{\frac{8gR_{vb}}{3}}$	1.0
	Numerical speed	$v_{turb} \approx \sqrt{\frac{8\cdot 10\cdot (1\div 3)\cdot 10^{-3}}{3}}[m/s] \approx (0.16\div 0.28)[m/s]$	0.1
	Ascending time	$t_{turb} > \frac{H}{v_{turb}} \approx \frac{0.1}{(0.16\div 0.28)}[s] \approx (0.35\div 0.62)[s]$	0.1
	TOTAL		10.0

Solutions:

S1. Consider a positive ion in the NaCl is surrounded by 26 neighbors (see Fig.1) . The first group of **6** *nearest* "central" neighbors have negative charges. They are located in distance **r** from the considered ion and their contribution is attractive:

$$V_{C0'}(r) = -6 \cdot k \frac{e^2}{r}$$

The next group of "central" neighbors are **12** ions standing in distance $\sqrt{r^2+r^2} = \sqrt{2}r$. All they have positive charge and act repulsively

$$V_{C0''}(r) = +\frac{12}{\sqrt{2}} \cdot k \frac{e^2}{r}$$

Easily to guess that the "vertex" **8** negative ions standing in distance $\sqrt{r^2+r^2+r^2} = \sqrt{3}r$ produce an attractive potential:

$$V_{C0'''}(r) = -\frac{8}{\sqrt{3}} \cdot k \frac{e^2}{r}$$

Having done summation of these contributions, we obtain the zero-order Coulomb potential

$$V_{C0}(r) = V_{C0'}(r) + V_{C0''}(r) + V_{C0'''}(r) = -\alpha_0 \cdot k \frac{e^2}{r}$$

where the approximate Madelung constant reads

$$\alpha_0 = 6 - \frac{12}{\sqrt{2}} + \frac{8}{\sqrt{3}} \approx 2.134$$

S2. The net potential energy is the sum of attractive and repulsive exponential parts and reads

$$V_1(r) = V_{att} + V_{rep1} = -\alpha \cdot k \frac{e^2}{r} + \lambda \cdot e^{-r/\rho}$$

The condition for the equilibrium position r= r_0 is

$$F(r_0) = \frac{dV_1}{dr}(r_0) = -\alpha \cdot k \frac{e^2}{r_0^2} + \frac{\lambda}{\rho} e^{-r_0/\rho} = 0$$

Then, r_0 may be defined from

$$e^{-r_0/\rho} = \rho \cdot \frac{\alpha \cdot k \cdot e^2}{\lambda \cdot r_0^2}$$

The equilibrium potential is:

$$V_1(r_0) = -\alpha \cdot k \frac{e^2}{r_0}\left(1 - \frac{\rho}{r_0}\right)$$

S3. The dissociation energy corresponding to an ion (pair of atoms) is

$$E_{pair} = \frac{E_{dis}}{N_A} = \frac{-764\ [kJ/mole]}{6.022 \cdot 10^{23} [1/mole]} \approx -1.269 \cdot 10^{-18} [J]$$

This energy will be spent to overcome the potential energy $E_{pair} = V_1(r_0) = -k \cdot \alpha \frac{e^2}{r_0}\left(1 - \frac{\rho}{r_0}\right)$

Then,

$$\frac{\rho}{r_0} = 1 + \frac{r_0 E_{pair}}{k\alpha e^2} \approx 1 - \frac{0.282 \cdot 10^{-9} \cdot 1.269 \cdot 10^{-18}}{9 \cdot 10^9 \cdot 1.7476 \cdot \left(1.6 \cdot 10^{-19}\right)^2} \approx 0.112...$$

Subsequently,

$$\rho = 0.112 \cdot r_0 \approx 0.0316 \cdot 10^{-9} [m]$$

This result is in good agreement with experimental data (see Table 2).

Crystal	r [nm]	ρ [nm]	E_{dis} $[kJ/mole]$
NaCl	0.282	0.032	-764.4
LiF	0.214	0.029	-1014.0
RbBr	0.345	0.034	-638.8

Table 2
Properties of Salt Crystals with the NaCl Structure [C.Kittel, "Introduction to Solid State Physics", N.Y., Wiley (1976) p.92]

S4. The repulsive inverse-power part leads to the following net potential

$$V_2(r) = V_{att} + V_{rep2} = -\alpha \cdot k \frac{e^2}{r} + \frac{b}{r^n}$$

The equilibrium position r=r_0 is defined from the zero-force condition

$$F(r_0) = \frac{dV_2(r_0)}{dr} = -\alpha \cdot k \frac{e^2}{r_0^2} - \frac{nb}{r_0^{n+1}} = 0$$

The equation for r_0 reads $r_0^{n-1} = \dfrac{b \cdot n}{\alpha \cdot k \cdot e^2}$

Then, $$V_2(r_0) = -\alpha \cdot k \frac{e^2}{r_0}\left(1 - \frac{1}{n}\right) = V_{Coulomb}(r_0) + V_{Pauli}(r_0)$$

Here, the first term corresponds to the Coulomb potential and the second – to the Pauli's.

S5. Comparing with previous result (see **3.** and **4.**) we express the Born exponent

$$n = \frac{r_0}{\rho} \approx \frac{1}{0.112} \approx 9$$

With n=9 one finds that

$$V_{Coulomb}(r_0):V_{Pauli}(r_0)=1:(1/9)$$

The Coulomb and the Pauli potentials contribute to the net potential with a proportion 9:1. This result agrees well with experimental data (see, Table 3).

Ion type	n
Na^+, F^+	7
K^+, Cu^+, Cl^-	9
Au^+, I^+	12

Table 3. The experimental fit for the Born exponent.
[CRC Handbook of Physics, 2004]

S6. The ionization energy of the **Na** atom is **+5.14 eV** while the electron affinity of the **Cl** atom
is **-3.61 eV**. Subsequently, the electron transfer energy per atom is the half difference

$$E_{trans} \approx \frac{+5.14-3.61}{2}[\text{eV}] \approx +0.77[\text{eV}]$$

The total binding energy per atom in the NaCl crystal is:

$$E_{bind} = \frac{E_{pair}}{2} + E_{trans} = \frac{-1.269\cdot 10^{-18}[\text{J}]}{2} + 1.232\cdot 10^{-19}[\text{J}] \approx \frac{-0.511\cdot 10^{-18}[\text{J}]}{1.602\cdot 10^{-19}[\text{J/eV}]} \approx -3.19[\text{eV}]$$

This estimate is in satisfactory agreement with the experimental result

$$E_{\exp} \approx -3.28[\text{eV}]$$

[*Marking Scheme*] Ionic Crystal, Yukawa-type Potential and Pauli Principle

Q	*Item*	*Answer*	*Points*
1.	*Attractive* Coulomb potential	$V_{C0}(r)=V_{C0'}(r)+V_{C0''}(r)+V_{C0'''}(r)=-\alpha_0\cdot k\frac{e^2}{r}$	*0.5*
	Madelung constant	$\alpha_0=6-\frac{12}{\sqrt{2}}+\frac{8}{\sqrt{3}}\approx 2.134$	*1.0*
2.	Equilibrium position	$F(r_0)=\frac{dV_1(r_0)}{dr}=-\alpha\cdot k\frac{e^2}{r_0^2}+\frac{\lambda}{\rho}e^{-r_0/\rho}=0$	*0.5*
	Net potential	$V_1(r_0)=-\alpha\cdot k\frac{e^2}{r_0}\left[1-\frac{\rho}{r_0}\right]$	*1.0*
3.	Pair energy	$E_{pair}=\frac{E_{dis}}{N_A}=\frac{\text{-764 }[kJ/mole]}{6.022\cdot 10^{22}[1/mole]}\approx -1.269\cdot 10^{-18}[J]$	*0.2*
	Equation for potential	$E_{pair}=V_1(r_0)=-k\alpha\frac{e^2}{r_0}\left[1-\frac{\rho}{r_0}\right]$	*0.9*
	Equation for range parameter	$\frac{\rho}{r_0}=1+\frac{r_0E_{pair}}{k\alpha e^2}\approx 1-\frac{0.282\cdot 10^{-9}\cdot 1.269\cdot 10^{-18}}{9\cdot 10^9\cdot 1.7476\cdot\left(1.6\cdot 10^{-19}\right)^2}\approx 0.112...$	*0.8*
	Range parameter	$\rho=0.112\cdot r_0\approx 0.0316\cdot 10^{-9}[m]$	*0.1*
4.	Equilibrium position	$F(r_0)=\frac{dV_2(r_0)}{dr}=-\alpha\cdot k\frac{e^2}{r_0^2}-\frac{nb}{r_0^{n+1}}=0$	*0.5*
	Equation for r_0	$r_0^{n-1}=\frac{b\cdot n}{\alpha\cdot k\cdot e^2}$	*0.5*
	Net potential	$V_2(r_0)=-\alpha\cdot k\frac{e^2}{r_0}\left[1-\frac{1}{n}\right]$	*1.0*
5.	Born exponent	$n=\frac{r_0}{\rho}\approx\frac{1}{0.112}\approx 9$	*1.2*
	Proportions	$V_{Coulomb}(r_0):V_{Pauli}(r_0)=1:(1/9)$	*0.3*
6.	Electron transfer energy	$E_{trans}\approx\frac{+5.14-3.61}{2}[\text{eV}]\approx +0.77[\text{eV}]$	*0.5*
	the total binding energy	$E_{bind}=\frac{E_{pair}}{2}+E_{trans}=\frac{\text{-}1.269\cdot 10^{-18}[\text{J}]}{2\cdot 1.602\cdot 10^{-19}[\text{J/eV}]}+0.77[\text{eV}]\simeq -3.19[\text{e}$	*1.0*
total			*10.0*

Problem 3. How does a superluminal object look like?

1. Radiating superluminal dot

(1) Expression of t in terms of d, t', u and v.

$t = t' + \sqrt{d^2 + (vt')^2}/u$	1.0

(2) The apparent position x'_0 in terms of d and θ.

$x'_0 = -d\cot\theta$	1.0

The observed time t_0 of the first appearance in terms of d, v and ϑ.

$t_0 = \frac{d}{v}\tan\theta$	1.0

(3) The apparent position(s) $x'(t)$ in terms of v, θ, t and t_0.

$x'_+ = v\cot^2\theta\left(-t + \cos^{-1}\theta\sqrt{t^2 - {t_0}^2}\right)$ $x'_- = v\cot^2\theta\left(-t - \cos^{-1}\theta\sqrt{t^2 - {t_0}^2}\right)$	2.0

(4) The apparent velocity(s) $v'(t)$ in terms of v, θ, t and t_0.

$v'_+(t) = v\cot^2\theta\{-1 + 1/[\cos\theta\sqrt{1-(t_0/t)^2}]\}$ along the $+x$ axis $v'_-(t) = v\cot^2\theta\{-1 - 1/[\cos\theta\sqrt{1-(t_0/t)^2}]\}$ along the $-x$ axis	1.0

(5) The apparent velocity(s) v' of the first appearance of the particle

$v'_+ = -\infty$ along the $+x$ axis $v'_- = \infty$ along the $-x$ axis	0.2

(6) The apparent velocity(s) v' of the particle at infinite distances in terms of v and u

$v'_+ = vu/(v+u) = u/(1+\cos\theta)$ along the $+x$ axis $v'_- = -vu/(v-u) = -u/(1-\cos\theta)$ along the $-x$ axis	0.2

(7) The graph of the apparent velocity v' versus time t. (Remember to write down the asymptotic values of the apparent velocity).

	1.0

(8) An apparent velocity CAN / CANNOT exceed the light speed in the vacuum. Circle the correct answer.

Can	0.2
Can not	

1.1. Radiating linear object

A. Parallel movement

(9) The time interval of complete appearance of the whole linear object from the first appearance of its front point. (in terms of L, γ and v)

$\Delta t = L/(\gamma v)$	0.3

(10) The apparent length(s) of the object at the moment of its complete appearance. (in terms of d, L, θ and γ)

$L_+ = \frac{L\cot^2\theta}{\gamma}\left(\cos^{-1}\theta\sqrt{1+\frac{2d\gamma\tan\theta}{L}}-1\right)$ $L_- = \frac{L\cot^2\theta}{\gamma}\left(\cos^{-1}\theta\sqrt{1+\frac{2d\gamma\tan\theta}{L}}+1\right)$	0.4

B. Perpendicular movement

(11) Show that the x and y coordinates of any given point of the object satisfy an elliptic equation

$$\frac{(x-x_c)^2}{a^2}+\frac{y^2}{b^2}=1$$	0.7

(12) The position x_c of the centre of symmetry of the ellipse in terms of v, t and θ.

$$x_c=-vt\cot^2\theta$$	0.5

(13) The lengths of the semi-major and semi-minor axes of the ellipse in terms of v, t and θ.

$$a=vt\frac{\cos\theta}{\sin^2\theta}$$ $$b=vt\cot\theta$$	0.5