Theoretical Question 1: Particles and Waves

SOLUTION

Part A. Inelastic scattering and compositeness of particles

(a) Let the momentum of the target particle after scattering be $\vec{P}$. The law of conservation of linear momentum implies $\vec{P} = \vec{p}_1 - \vec{p}_2$. The total translational kinetic energies of the scattering system before and after scattering are, respectively,

$$K_i = \frac{{p_1}^2}{2m}$$

$$K_f = \frac{{p_2}^2}{2m} + \frac{(\vec{p}_1 - \vec{p}_2)^2}{2M} = \frac{{p_2}^2}{2m} + \frac{1}{2M}({p_1}^2 - 2p_1 p_{2x} + {p_2}^2). \qquad \text{(a-1)}$$

(i) By definition, we have $Q = K_i - K_f$, or equivalently,

$$Q = K_i - K_f = \frac{{p_1}^2}{2m} - \frac{{p_2}^2}{2m} - \frac{1}{2M}({p_1}^2 - 2p_1 p_{2x} + {p_2}^2)$$

$$= \frac{1}{2mM}\{(M-m){p_1}^2 - (M+m){p_2}^2 + 2m p_1 p_{2x}\}$$

$$= \frac{M+m}{2mM}\{\frac{M-m}{M+m}{p_1}^2 - {p_2}^2 + \frac{2m}{M+m}p_1 p_{2x}\}$$

$$= \frac{M+m}{2mM}\{\frac{M-m}{M+m}{p_1}^2 - {p_{2x}}^2 + \frac{2m}{M+m}p_1 p_{2x} - {p_{2y}}^2\}$$

$$= \frac{M+m}{2mM}\{[\frac{M-m}{M+m} + (\frac{m}{M+m})^2]{p_1}^2 - (p_{2x} - \frac{m}{M+m}p_1)^2 - {p_{2y}}^2\}$$

$$= \frac{M+m}{2mM}\{(\frac{M}{M+m})^2 {p_1}^2 - (p_{2x} - \frac{m}{M+m}p_1)^2 - {p_{2y}}^2\} \qquad \text{(a-2)*}$$

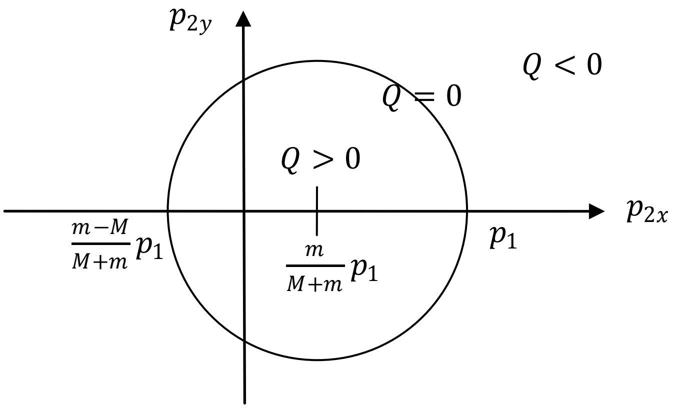
(ii) If the incident and target particles are both *elementary*, their internal energies remain the same before and after the scattering. By the law of conservation of energy, we must have $K_i = K_f$, or $Q = 0$. Thus, we obtain from Eq. (a-2) the following equality

$$(\frac{M}{M+m})^2 {p_1}^2 = (p_{2x} - \frac{m}{M+m}p_1)^2 + {p_{2y}}^2$$

In the p_{2x}-p_{2y} plane, this represents a circle centered at $(mp_1/(M+m), 0)$ with radius $Mp_1/(M+m)$. The case $m < M$ is shown in Fig. A1. The values of p_{2x} at the intercepts of the circle with the p_{2x}-axis are

$$\frac{m}{M+m}p_1 - \frac{M}{M+m}p_1 = \frac{m-M}{M+m}p_1 \quad \text{and} \quad \frac{m}{M+m}p_1 + \frac{M}{M+m}p_1 = p_1. \qquad \text{(a-3)*}$$

Fig. A1



For a *composite* target in its ground state before scattering, the law of conservation of energy implies

$$K_{\rm i} = K_{\rm f} + \Delta E_{\rm int},$$

where $\Delta E_{\rm int} \geq 0$ is the change in internal energy (or excitation energy) of the target as a result of scattering and $K_{\rm i}$ and $K_{\rm f}$ are given by Eq. (a-1). Thus, in this case, the total translational kinetic energy loss is given by $Q = K_{\rm i} - K_{\rm f} = \Delta E_{int} \geq 0$. For points on the circumference of the circle in Fig. A1, we have $Q = 0$, i.e. elastic scattering. For the interior of the circle, we have $Q > 0$, corresponding to inelastic scattering with the target in an excited state after scattering.

The circle and its interior ($Q \geq 0$) are thus allowed by a composite target in its ground state before scattering.

(b) Let L be the angular momentum of the target about an axis through its center of mass and normal to the plane of particle motions after scattering. By the law of conservation of angular momentum,

$$L = \pm(\tfrac{1}{2} d_0 \sin\theta)(p_1 - p_2) \qquad \text{(a-4)}$$

where the + (or –) sign is implied if the target particle on the left (or right) in Fig. 2 is hit by the incident particle.

(i) After scattering, the target may undergo vibrational and rotational motions. When the spring reaches its maximum extension, the length of the spring is $d_{\rm m} = (1+x)d_0$ and the moment of inertia of the target rotating about an axis through its center of mass and perpendicular to the spring is $I_{\rm m} = \frac{1}{4}M{d_{\rm m}}^2 = \frac{1}{4}M{d_0}^2(1+x)^2$. The law of conservation of energy implies

$$Q = \frac{1}{2}k(d_{\rm m} - d_0)^2 + \frac{L^2}{2I_{\rm m}} \qquad \text{(a-5)}$$

where the last term represents the rotational kinetic energy of the target at the maximum extension of the spring. According to Eq. (a-4), we have

$$\frac{L^2}{2I_{\mathrm{m}}} = (\frac{d_0}{d_{\mathrm{m}}})^2 \frac{(p_1 - p_2)^2}{2M} \sin^2\theta = (\frac{1}{1+x})^2 \frac{(p_1 - p_2)^2}{2M} \sin^2\theta \qquad \text{(a-6)}$$

and therefore

$$Q = \frac{1}{2} k {d_0}^2 x^2 + (\frac{1}{1+x})^2 \frac{(p_1 - p_2)^2}{2M} \sin^2\theta \,. \qquad \text{(a-7)*}$$

Note that, since $p_{2y} = 0$ and $p_{2x} \equiv p_2$, we have from Eq. (a-2)

$$Q = K_{\mathrm{i}} - K_{\mathrm{f}} = \frac{M+m}{2mM}\{(\frac{M}{M+m})^2 {p_1}^2 - (p_2 - \frac{m}{M+m}p_1)^2\}$$

$$= \frac{(p_1 - p_2)}{2mM}\{(M-m)p_1 + (M+m)p_2\} \qquad \text{(a-8)}$$

A scattering can occur only if $p_1 \neq p_2$ and $Q \geq 0$, so from Eq. (a-8), we obtain

$$-\frac{M-m}{M+m} p_1 \leq p_2 < p_1 \qquad \text{(a-9)*}$$

where the equalities hold only if $Q = 0$.

**An equation marked with an asterisk gives key answers to the problem.*

(ii) The scattering cross section σ is given by the numerical range of $\alpha = \sin^2\theta$. For given p_1 and p_2, Q is a constant by Eq. (a-8), and the value of α can be found from Eq. (a-7) to be

$$\alpha = \sin^2\theta = \frac{2M}{(p_1 - p_2)^2}(1+x)^2 (Q - \frac{1}{2} k {d_0}^2 x^2) \geq 0.$$

In the limit of large k, the last inequality can hold only if x is very small. Thus x may be neglected in the factor $(1 + x)$ and we obtain

$$\alpha = \sin^2\theta \approx \frac{2M}{(p_1 - p_2)^2}(Q - \frac{1}{2} k {d_0}^2 x^2) = \beta (1 - \frac{1}{2Q} k {d_0}^2 x^2), \qquad \text{(a-10)}$$

where

$$\beta \equiv \frac{2MQ}{(p_1 - p_2)^2} = \frac{1}{m(p_1 - p_2)}\{(M-m)p_1 + (M+m)p_2\} \qquad \text{(a-11)}$$

is nonnegative and the last equality follows from Eq.(a-8).

From Eq. (a-10), the minimum value $\alpha_{\min}$ of $\alpha = \sin^2\theta$ is found to be zero for all $Q \geq 0$, and this occurs when

$$x^2 = \frac{2Q}{k {d_0}^2} \qquad (\alpha_{\min} = 0).$$

Moreover, the maximum value $\alpha_{\max}$ of $\alpha = \sin^2\theta$ is seen to be given by

$$\alpha_{\max} = \begin{cases} \beta & \text{if } \beta \le 1 \text{ and } x = 0, \\ 1 & \text{if } \beta \ge 1 \text{ and } \left(1 - \dfrac{1}{2Q} k {d_0}^2 x^2\right) = \dfrac{1}{\beta}. \end{cases} \tag{a-12}$$

Note that, from Eq. (a-11), it follows

$$\beta = \begin{cases} \le 1 & \text{if } p_2 \le -\dfrac{M-2m}{M+2m} p_1, \\ \ge 1 & \text{if } p_2 \ge -\dfrac{M-2m}{M+2m} p_1. \end{cases}$$

Since $\alpha_{\min} = 0$, the cross section is given by $\sigma = \alpha_{\max} - \alpha_{\min} = \alpha_{\max}$ and, from Eq. (a-12), we see that it becomes 1 and is independent of p_2 when $\beta \ge 1$. Thus the threshold value p_c at which scaling of cross section starts is given by

$$p_c = -\frac{M-2m}{M+2m} p_1. \tag{a-13)*$$

For p_2 below the threshold value, the cross section is equal to β according to Eq. (a-12) and, from Eq. (a-11), we have the following result

$$\sigma = \beta = \frac{(M-m)p_1 + (M+m)p_2}{m(p_1 - p_2)} = \frac{M(p_1 + p_2)}{m(p_1 - p_2)} - 1. \quad (p_2 \le p_c) \tag{a-14}$$

Note that, from Eq. (a-9), we have $\sigma \ge 0$ for $p_2 \le p_c$ as expected. The cross section σ as a function of p_2 is shown in Fig. A2 which evidently shows scaling behavior.

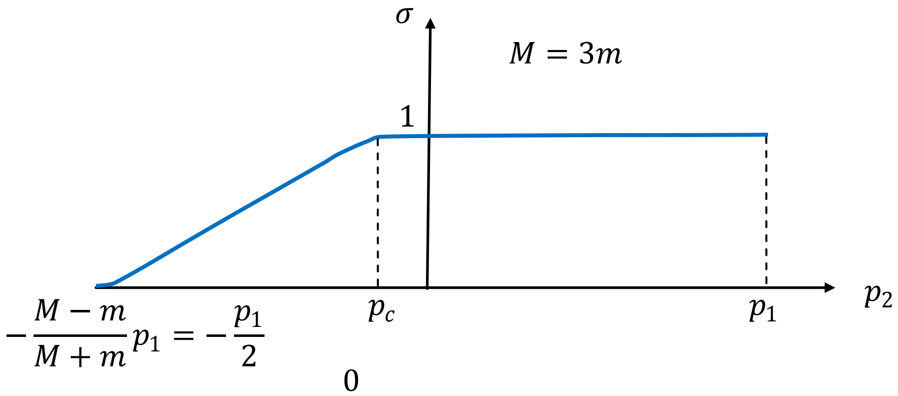


Figure A2

Part B. Waves on a string

(c) The initial disturbance will propagate toward the fixed ends. It can be considered as the superposition of two waves with wave forms $y_R(x-ct)$ and $y_L(x+ct)$, travelling toward the right and the left, respectively. They will both be reflected out of phase at the fixed ends. At $t=0$, the sum of their displacements must be equal to the initial wave form $f(x)$. Therefore

$$y_R(x)+y_L(x)=f(x), \quad 0\le x\le L \qquad \text{(b-1)}$$

Let $y'(x)=dy/dx$. At $t=0$, the string is at rest and the sum of velocities $\dot{y}_R=-cy'_R$ and $\dot{y}_L=cy'_L$ of the two waves at x must be zero. Thus we have

$$y'_R(x)-y'_L(x)=0, \qquad 0\le x\le L \qquad \text{(b-2)}$$

Integrating Eq. (b-2) with respect to x and combining with Eq. (b-1), we obtain

$$y_R(x)=\frac{1}{2}(f(x)+\ y_0\ ), \qquad y_L(x)=\frac{1}{2}(f(x)-\ y_0), \ \le x\le L, \quad \text{(b-3)}$$

where y_0 is a constant. (Note that this result may also be obtained by graphical construction which takes initial conditions into account by superposing two pulses of identical wave form but travelling in opposite directions.)

Since both waves, after being reflected once at each end and having travelled a distance $2L$, return to its original position and state, the period is thus

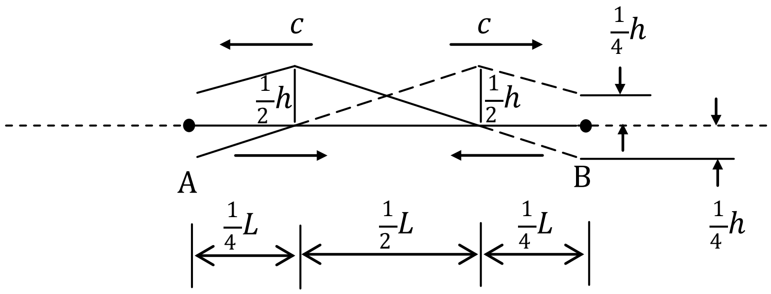
$$T=\frac{2L}{c}. \qquad \text{(b-4)*}$$

[Another way to get the period T]:

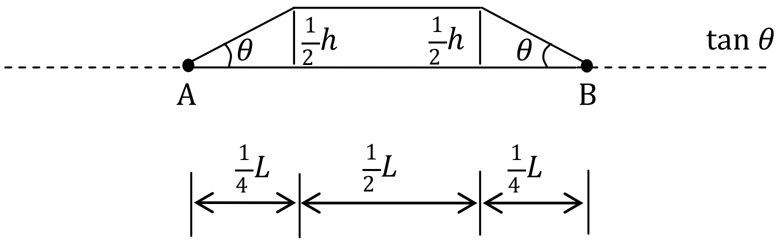
Because the first harmonics (with the longest period) has $\lambda=2L$, we have

$$T=\frac{1}{f}=\frac{\lambda}{c}=\frac{2L}{c}. \qquad \text{(b-4')*}$$

At $t=T/8$, 1/8 of each wave will have been reflected out of phase as shown below, where the result for the wave traveling to the right is shown as dashed lines,



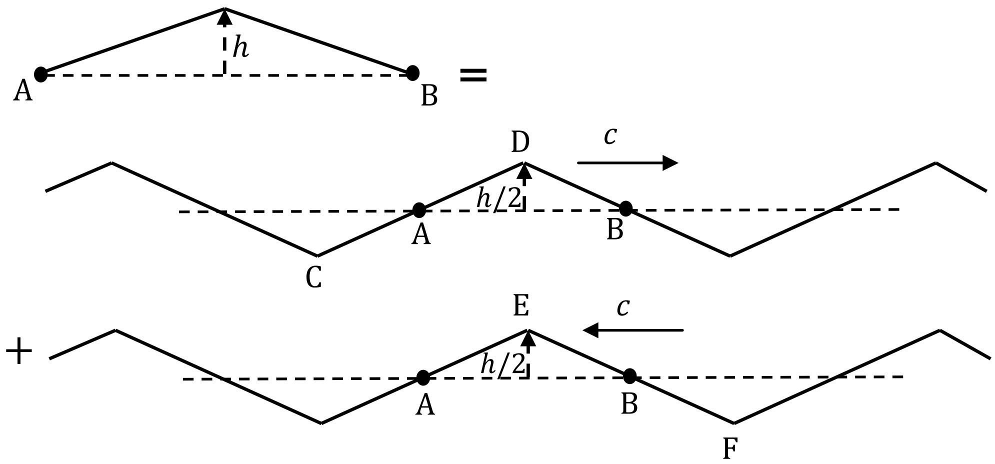
The displacements of all wave components may be added to give the resultant wave form at $t = T/8$, as shown below.



$\tan\theta = 2h/L$ (b-5)*

[Another solution of (c)]:

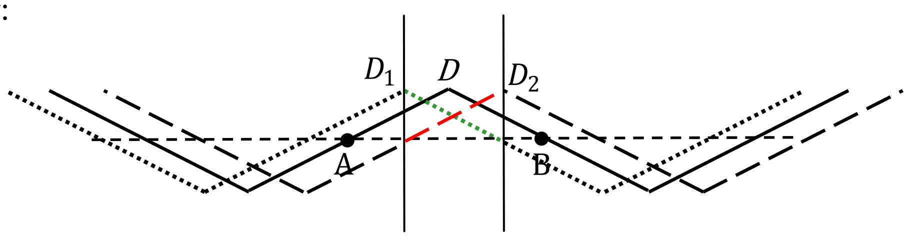
Since the evolution of the string is periodic, waves that can simulate it must be periodic in space. Furthermore, since the string is released from rest, we can consider the initial configuration of the string as a superposition of two saw-tooth waves travelling in opposite directions as shown below:



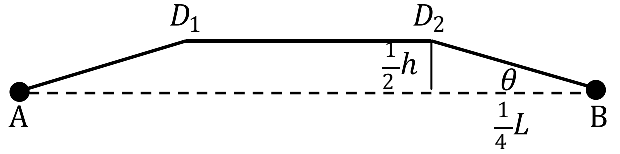
Both A and B will be fixed because two saw-tooth waves tend to move A or B in opposite directions. Clearly, the period of motion is the time for the saw-tooth wave to travel the distance $2L$. Hence we obtain

$$T = \frac{2L}{c}. \qquad \text{(b-4'')*}$$

At any time t, the shape of the string is determined by adding up the two waves as shown below:



From the figure above, one sees that between D_1 and D_2, two saw-tooth waves (line marked by green and red line) have opposite slopes and hence their sum between D_1 and D_2 is constant with the height being given by $h/2$ (the height of D_1 or D_2). Between A and D_1 or B and D_2, two saw-tooth waves have the same slope but move in opposite direction, hence their sum simply reproduces the original saw-tooth shape. Thus, the shape of string at time t is



with $\overline{D_1D_2} = 2ct$. For $t = T/8$, $\overline{D_1D_2} = L/2$ and

$$\tan\theta = \frac{2h}{L}. \qquad \text{(b-5'')}^*$$

(d) To find the total energy, we note that the normal force F which pulls the string sideways at the midpoint is

$$F(y) = 2\tau\sin\theta = 2\tau\frac{2y}{L}, \qquad \text{(b-6)}$$

where τ is the constant tension ($h \ll L$) on the string and y is the transverse displacement at the midpoint. The work done by F is the total mechanical energy given to the string or

$$E = \int_0^h F(y)\mathrm{d}y = 2\tau\frac{h^2}{L} = 2\mu c^2\frac{h^2}{L}, \qquad \text{(b-7)*}$$

where use has been made of $c = \sqrt{\tau/\mu}$.

[Another solution of (d)]:

Because $\mathrm{c} = \sqrt{\tau/\mu}$ with τ being the tension on the string, we have $\tau = \mu c^2$. The total mechanical energy E at $t = 0$ is the potential energy

$$E = U = \frac{1}{2}\tau\int_0^L\left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{1}{2}\mu c^2\left(\frac{h}{L/2}\right)^2 L = 2\mu c^2\frac{h^2}{L} \qquad \text{(b-7')}^*$$

Here $y(x,t)$ is the displacement of the elastic string.

[Yet another solution of (d)]:

We consider a special moment when $\overline{CD}$ and $\overline{EF}$ move into $\overline{AB}$ region completely. At this moment, the string is flat so that total mechanical energy is equal to the total kinetic energy. Since the velocity for each point on the string is $2c\tan\alpha$ (downward) with $\tan\alpha = h/L$, we obtain

$$E = \frac{1}{2}\mu L\,(2c\tan\alpha)^2 = 2\mu c^2\frac{h^2}{L}. \qquad \text{(b-7'')*}$$

Part C. The expanding universe

(e) The photons were emitted at t_e, and are received now at t_0, so

$$\frac{a(t_0)}{a(t_e)} = \frac{\lambda(t_0)}{\lambda(t_e)} = \frac{145.8}{121.5} \approx 1.200 \tag{c-1}$$

On the other hand, the Hubble parameter can be derived as

$$a(t) \propto \exp(bt) \rightarrow H(t) = \frac{\dot{a}(t)}{a(t)} = b, \tag{c-2}$$

which is independent of time.

Within dt at some moment t in the past, the photons traveled cdt, which was

$$\frac{a(t_e)}{a(t)} cdt$$

at time t_e due to the cosmic expansion. The photons were emitted at t_e so the distance of the star from us at that time is

$$L(t_e) = \int_{t_e}^{t_0} \frac{a(t_e)}{a(t)} cdt = c \int_{t_e}^{t_0} \exp[H(t_e - t)]\, dt$$

$$= \frac{c}{H}(1 - \exp[H(t_e - t_0)]). \tag{c-3}$$

We already know from Eq. (c-2) that

$$\frac{\exp(Ht_0)}{\exp(Ht_e)} = \frac{a(t_0)}{a(t_e)} \approx 1.200, \tag{c-4}$$

thus

$$L(t_e) = \frac{c}{H}\left(1 - \frac{1}{1.200}\right) \approx 690 \text{ Mpc.} \tag{c-5}^*$$

(f) Due to the cosmic expansion, the above distance is actually longer now:

$$L(t_0) = \frac{a(t_0)}{a(t_e)} L(t_e) = \frac{a(t_0)}{a(t_e)} \frac{c}{H}\left(1 - \frac{a(t_e)}{a(t_0)}\right) \tag{c-6}$$

Thus according to the Hubble Law, we can compute the receding velocity of the star now:

$$\mathrm{v}(t_0) = HL(t_0) = H \frac{a(t_0)}{a(t_e)} \frac{c}{H}\left(1 - \frac{a(t_e)}{a(t_0)}\right) = \left(\frac{a(t_0)}{a(t_e)} - 1\right) c \approx 0.200\, c \tag{c-7}^*$$

Theoretical Question 1: Particles and Waves

MARKING SCHEME

Total	Mark(s)	Marking Scheme for Answers
Part A 4.0	(a) 1.1	(i) Q in terms of m, M, p_1, p_{2x}, and p_{2y}. ➢ 0.2 for expression of Q → (a-2)† (ii) **Plot** of condition relating p_1, p_{2x}, and p_{2y} . ➢ 0.2 for circle and the position of its center ➢ 0.1 for intersection point $(m-M)p_1/(m+M)$ → (a-3) ➢ 0.1 for intersection point p_1 → (a-3) ➢ 0.3 for labeling regions for $Q=0$, $Q>0$, and $Q<0$ (0.1 each) **Allowed regions of** Q. ➢ 0.2 for allowed regions: $Q>0$ and $Q=0$ (0.1 each)
	(b) 2.9	(i) Equation relating x to $Q, \theta, d_0, m, k, M, p_1$ and p_2. ➢ 0.2 for correctly stating the energy conservation → (a-5) ➢ 0.2 for correct rotational energy expression → (a-6) ➢ 0.3 for expression of Q → (a-7) (ii) **Threshold value** p_c in terms of m, M, and p_1. ➢ 0.3 for $\alpha_{\min}=0$ ➢ 0.4 for $\alpha_{\max}$ → (a-12) ➢ 0.4 for expression of p_c. → (a-13) **Sketch of** σ **versus** p_2. ➢ 0.4 for σ increasing with p_2 quasi-linearly and becoming level beyond p_c →(a-14) ➢ 0.4 for range of p_2 →(a-9) ➢ 0.3 for range of $\sigma=(0,1)$
Part B 3.0	(c) 2.2	**Period of vibration** T. ➢ 0.5 for $T=2L/c$ →(b-4) **Shape of the string** at $t=T/8$. ➢ 0.5 for decomposing the triangle into two traveling waves ➢ 0.5 for correct shape →(b-5) ➢ 0.3 for correct lengths $L/4, L/2$ and $L/4$ ➢ 0.2 for correct height $h/2$ ➢ 0.2 for $\tan\theta=2h/L$

	(d) 0.8	The total mechanical energy E. ➢ 0.4 for expression of $E = 2\mu h^2 c^2/L$ (for all cases below) →(b-7) For the remaining 0.4 point: case 1: calculating the work done by normal force ➢ 0.2 for correct expression of the normal force →(b-6) ➢ 0.2 for correct relation of E to the normal force or case 2: calculating the potential energy ➢ 0.4 for correct form of the potential energy →(b-7') or case 3: calculating the kinetic energy ➢ 0.4 for calculating velocity correctly →(b-7")
Part C $\boxed{3.0}$	(e) 2.2	Distance (in units of Mpc) of the star from us. ➢ 1.0 for $L(t_e) = \int_{t_e}^{t_0} \frac{a(t_e)}{a(t)} c dt$ →(c-3) ➢ 0.5 for $L(t_e) = \frac{c}{H}(1 - \exp[H(t_e - t_0)])$ →(c-3) ➢ 0.4 for $\exp[H(t_0 - t_e)] \approx 1.200$ →(c-4) ➢ 0.3 for value of $L(t_e) \approx 690$ Mpc →(c-5)
	(f) 0.8	The receding velocity (in units of c) of the star. ➢ 0.3 for $L(t_0) = \frac{a(t_0)}{a(t_e)} L(t_e)$ or $L(t_0) = \frac{\lambda(t_0)}{\lambda(t_e)} L(t_e)$ →(c-5) ➢ 0.2 for expression of $v(t_0)$ →(c-7) ➢ 0.3 for value of $v(t_0) \approx 0.200\ c$ →(c-7)

†The equation number(s) at the end of a line refers to equation(s) in the SOLUTION sheets.

Theoretical Question 2: Strong Resistive Electromagnets

SOLUTION

Part A. Magnetic Fields on the Axis of the Coil

(a) At the point x on the axis, the magnetic field due to the current I passing through the turns located in the interval $(s, s+ds)$ is (see Fig. A1)

$$d\vec{B} = \left(\frac{\mu_0}{4\pi}\right)\frac{I(\pi D)}{(D/2)^2+(s-x)^2}\cdot\frac{(D/2)}{\sqrt{(D/2)^2+(s-x)^2}}\cdot\frac{ds}{a}\hat{x} \qquad \text{(a-1)}$$

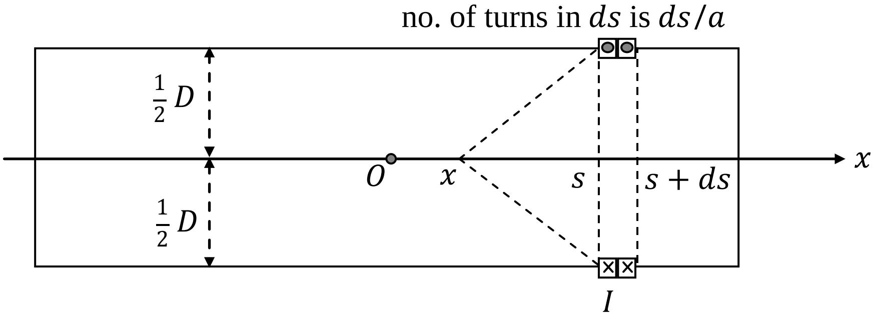
which, when summed over all turns of the coil, leads to the total magnetic field $\vec{B}(x) = B(x)\hat{x}$ with

$$B(x) = \frac{\mu_0 I}{2a}\left(\frac{D}{2}\right)^2\int_{-\ell/2}^{\ell/2}\frac{ds}{[(D/2)^2+(s-x)^2]^{3/2}}$$

$$= \frac{\mu_0 I}{2a}\left(\frac{D}{2}\right)^2\int_{-\ell/2-x}^{\ell/2-x}\frac{ds}{[(D/2)^2+s^2]^{3/2}}$$

$$= \frac{\mu_0 I}{2a}\left\{\frac{(\ell/2)-x}{\sqrt{(D/2)^2+[(\ell/2)-x]^2}}+\frac{(\ell/2)+x}{\sqrt{(D/2)^2+[(\ell/2)+x]^2}}\right\} \qquad \text{(a-2)*}$$

Figure A1



(b) From Eq. (a-2), the magnetic field at O wth $x = 0$ is

$$B(0) = \frac{\mu_0 I}{2a}\frac{2(\ell/2)}{\sqrt{(D/2)^2+(\ell/2)^2}} = \frac{\mu_0 I}{a}\frac{1}{\sqrt{1+(D/\ell)^2}} \qquad \text{(b-1)}$$

If $B(0)$ is 10.0 T, then the current I must be equal to

$$I_0 = B(0)\frac{a}{\mu_0}\sqrt{1+(D/\ell)^2} = 1.7794\times10^4\ \text{A} \cong 1.8\times10^4\ \text{A} \qquad \text{(b-2)*}$$

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*An equation marked with an asterisk gives key answers to the problem.

Part B. The Upper Limit of Current

(c) For an infinitely long coil with $\ell \to \infty$ and $b \ll D$, the magnetic field $\vec{B}$ acting on the current is the average of the fields inside and outside of the coil. The field outside is zero and the field inside is the same as that at O, i.e. $B(0)$ in Eq. (b-1) with $\ell \to \infty$. Thus we have

$$\vec{B} = \bar{B}\hat{x} = \frac{1}{2}(0 + \frac{\mu_0 I}{a})\hat{x} = \frac{\mu_0 I}{2a}\hat{x}, \tag{c-1}$$

and the outward normal force on the wire segment of length Δs is

$$\Delta F_{\mathrm{n}} = I\bar{B}\Delta s = I\Delta s\left(\frac{\mu_0 I}{2a}\right) \quad \text{or} \quad \frac{\Delta F_{\mathrm{n}}}{\Delta s} = \frac{\mu_0}{2a}I^2. \qquad \text{(c-2)*}$$

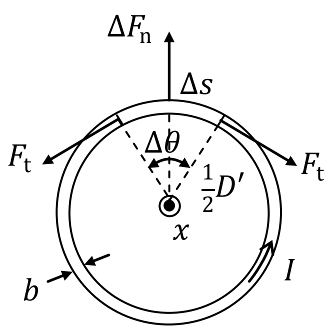
As can be seen from Fig. A2, the resultant of the pair of tension forces at the ends of the segment Δs is given by

$$-2F_{\mathrm{t}}\ \sin\left(\frac{\Delta\theta}{2}\right) \cong -F_{\mathrm{t}}\,\Delta\theta = -F_{\mathrm{t}}\left(\frac{2\Delta s}{D'}\right). \tag{c-3}$$

This must be in equilibrium with the normal force ΔF_{n} so that, by using Eq. (c-2), we have

$$\Delta F_{\mathrm{n}} = F_{\mathrm{t}}\left(\frac{2\Delta s}{D'}\right) \quad \text{or} \quad F_{\mathrm{t}} = \frac{D'}{2}\left(\frac{\Delta F_{\mathrm{n}}}{\Delta s}\right) = \frac{\mu_0}{4a}I^2 D'. \qquad \text{(c-4)*}$$

Figure A2



(d) At breaking, the tensile stress of the wire is, from Eq. (c-4),

$$\frac{F_{\mathrm{t}}}{ab} = \frac{\mu_0}{4a^2 b}I_{\mathrm{b}}{}^2 D' = \sigma_{\mathrm{b}} = 4.55 \times 10^8\ \mathrm{Pa}, \tag{d-1}$$

and the tensile strain of the wire is

$$\frac{\pi(D'-D)}{\pi D} = \frac{(D'-D)}{D} = 60\ \% \quad \text{or} \quad D' = 1.60\,D. \tag{d-2}$$

From the last two equations, the current I_{b} at which the turn will break is

$$I_{\mathrm{b}} = 2a\sqrt{\frac{b\sigma_{\mathrm{b}}}{\mu_0 D'}} = 2a\sqrt{\frac{b\sigma_{\mathrm{b}}}{\mu_0(1.60\,D)}} = 1.737 \times 10^4\ \mathrm{A} \cong 1.7 \times 10^4\ \mathrm{A}, \tag{d-3}$$

and the magnitude of the magnetic field at the center O, i.e. Eq. (b-1) with $\ell \to \infty$, is

$$B_{\mathrm{b}} = \frac{\mu_0 I_{\mathrm{b}}}{a} = 2\sqrt{\frac{\mu_0 b\sigma_{\mathrm{b}}}{D'}} = 10.914\ \mathrm{T} = 1.1 \times 10^1\ \mathrm{T}, \qquad \text{(d-4)*}$$

Part C. Rate of Temperature Rise

(e) When the current I is 10.0 kA, the current density J is given by

$$J = \frac{I}{ab} = \frac{1.00 \times 10^4}{(2.0 \times 10^{-3})(5.0 \times 10^{-3})} = 1.0 \times 10^9 \text{ A/m}^2. \tag{e-1}$$

The power density is given by

$$\rho_e J^2 = \rho_e \left(\frac{I}{ab}\right)^2 = 1.720 \times 10^{10} \text{ W/m}^3 \cong 1.7 \times 10^{10} \text{ W/m}^3. \qquad \text{(e-2)*}$$

(ALTERNATIVE)

The volume τ and resistance R (appearing also in Problem (h)) of the current-carrying wire for a coil of length ℓ are given by

$$\tau = \pi\left\{\left(\frac{D+b}{2}\right)^2 - \left(\frac{D-b}{2}\right)^2\right\}\ell = \pi bD\ell = N\pi abD, \tag{e-3}$$

$$R = \rho_e \frac{N\pi D}{ab} = \rho_e \frac{\pi D\ell}{a^2 b} = 1.9453 \times 10^{-2}\ \Omega \cong 1.9 \times 10^{-2}\ \Omega. \tag{e-4}$$

The total power P of Joule heat generated in the coil is

$$P = I^2 R = 1.9453 \times 10^6 \text{ W} = 1.9 \times 10^6 \text{ W}. \tag{e-5}$$

Thus the power density is

$$\frac{P}{\tau} = \frac{P}{N\pi abD} = \frac{P}{\ell\pi bD} = 1.7 \times 10^{10} \text{ W/m}^3. \qquad \text{(e-6)*}$$

Note that, by Eqs. (e-3) to (e-5), the expression for power density may also be written as

$$\frac{P}{\tau} = \frac{I^2 R}{\tau} = \frac{I^2}{\ell\pi bD}\rho_e \frac{\pi D\ell}{a^2 b} = \rho_e \left(\frac{I}{ab}\right)^2 = \rho_e J^2. \qquad \text{(e-7)*}$$

This is identical to that obtained in Eq. (e-2).

(f) The time rate of temperature increase of the coil is

$$\dot{T} = \frac{\rho_e J^2}{\rho_m c_p} = \frac{\rho_e}{\rho_m c_p}\left(\frac{I}{ab}\right)^2. \tag{f-1}$$

At $T = 293$ K and $I = 10.0$ kA, we have

$$\dot{T} = \frac{\rho_e}{\rho_m c_p}\left(\frac{I}{ab}\right)^2 = \frac{\rho_e J^2}{\rho_m c_p} = 4.975 \times 10^3 \text{ K/s} \cong 5.0 \times 10^3 \text{ K/s}. \qquad \text{(f-2)*}$$

(ALTERNATIVE)

The heat capacity of the coil is

$$Mc_p = \rho_m(\ell\pi bD)c_p = 3.9101 \times 10^2 \text{ J/K} \cong 3.9 \times 10^2 \text{ J/K}. \tag{f-3}$$

From Eqs. (e-5) and (f-3), the time rate of temperature increase is

$$\dot{T} = \frac{I^2 R}{Mc_p} = 4.975 \times 10^3 \text{ K/s} \cong 5.0 \times 10^3 \text{ K/s}. \qquad \text{(f-4)*}$$

Part D. A Pulsed-Field Magnet

(g) The magnetic flux ϕ_B through each turn is, in the limit $\ell \to \infty$, given by

$$\phi_B = \{\lim_{\ell\to\infty} B(0)\}\pi\left(\frac{D}{2}\right)^2 = \frac{\mu_0 I}{a}\pi\left(\frac{D}{2}\right)^2. \qquad \text{(g-1)}$$

The inductance L of the coil is

$$L = \frac{N\phi_B}{I} = \frac{N\mu_0}{a}\pi\left(\frac{D}{2}\right)^2 = \frac{\ell\mu_0}{4a^2}\pi D^2 = 1.0659\times10^{-4}\ \text{H} \cong 1.1\times10^{-4}\ \text{H}. \qquad \text{(g-2)*}$$

The resistance R of the coil is the same as given in Eq. (e-4). Thus

$$R = \rho_{\text{e}}\frac{\pi DN}{ab} = \rho_{\text{e}}\frac{\pi D\ell}{a^2 b} = 1.9453\times10^{-2}\ \Omega \cong 1.9\times10^{-2}\ \Omega. \qquad \text{(g-3)*}$$

(h) According to Kirchhoff's circuit law, the change of electric potential around a closed circuit must be zero and we have

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = 0. \qquad \text{(h-1)}$$

In this question, we are given

$$Q(t) = \frac{CV_0}{\sin\theta_0}e^{-\alpha t}\sin(\omega t+\theta_0) = \left[e^{\alpha\left(\frac{\theta_0}{\omega}\right)}\right]\frac{CV_0}{\sin\theta_0}e^{-\alpha\left(t+\frac{\theta_0}{\omega}\right)}\sin\omega\left(t+\frac{\theta_0}{\omega}\right), \qquad (1)$$

$$I(t) = \frac{dQ}{dt} = \left[\left(\frac{-\alpha}{\cos\theta_0}\right)\right]\frac{CV_0}{\sin\theta_0}e^{-\alpha t}\sin\omega t, \qquad (2)$$

$$\tan\theta_0 = \frac{\omega}{\alpha}. \qquad (3)$$

Comparing the right sides of Eqs. (1) and (2), one sees that the current $I(t) = dQ/dt$ is obtained from $Q(t)$ by changing the latter's time variable t to $(t-\theta_0/\omega)$ or, equivalently, changing $(t+\theta_0/\omega)$ to t, and then multiplying its amplitude constant by a factor

$$\left\{e^{-\alpha\frac{\theta_0}{\omega}}\left(\frac{-\alpha}{\cos\theta_0}\right)\right\}.$$

Since $I(t)$ in Eq. (2) has the same form as $Q(t)$ in Eq. (1), we may apply the same rule again to obtain its derivative dI/dt as

$$\begin{aligned}\frac{dI}{dt} &= \left\{e^{-\alpha\frac{\theta_0}{\omega}}\left(\frac{-\alpha}{\cos\theta_0}\right)\right\}\left[\left(\frac{-\alpha}{\cos\theta_0}\right)\frac{CV_0}{\sin\theta_0}\right]e^{-\alpha\left(t-\frac{\theta_0}{\omega}\right)}\sin\omega\left(t-\frac{\theta_0}{\omega}\right)\\ &= \left(\frac{\alpha}{\cos\theta_0}\right)^2\frac{CV_0}{\sin\theta_0}e^{-\alpha t}\sin(\omega t-\theta_0) \qquad \text{(h-2)}\end{aligned}$$

Making use of Formula 2 given in Appendix, we may express the left side of Eq. (h-1) as a linear combination of $\cos\omega t$ and $\sin\omega t$ so that

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = \left(\frac{CV_0}{\sin\theta_0}\right)e^{-\alpha t}(A\cos\theta_0\sin\omega t + B\sin\theta_0\cos\omega t) = 0, \qquad \text{(h-3)}$$

which can be satisfied if and only if

$$A \equiv L\left(\frac{\alpha}{\cos\theta_0}\right)^2 - R\left(\frac{\alpha}{\cos^2\theta_0}\right) + \frac{1}{C} = 0, \tag{h-4}$$

$$B \equiv -L\left(\frac{\alpha}{\cos\theta_0}\right)^2 + \frac{1}{C} = 0, \tag{h-5}$$

Note that Eqs. (h-4) and (h-5) may be obtained more simply by considering Eq. (h-1) at the moments when $\sin\omega t = 1$ and 0, respectively. Subtracting Eq. (h-5) from Eq. (h-4), we obtain

$$\alpha = \frac{R}{2L}, \tag{h-6}$$

If we use the expressions given in Eqs. (g-2) and (g-3), we obtain

$$\alpha = \frac{R}{2L} = \frac{\rho_e \dfrac{\pi D \ell}{a^2 b}}{\pi D^2 \dfrac{\ell \mu_0}{2a^2}} = \frac{2\rho_e}{\mu_0 b D} = 9.1249 \times 10^1\ \mathrm{s}^{-1} \cong 9.1 \times 10^1\ \mathrm{s}^{-1}. \tag{h-7}*$$

Adding up Eqs. (h-4) and (h-5), we have, by Eq. (h-6) and Eq. (3),

$$\frac{1}{LC} = \frac{R\alpha}{2L\cos^2\theta_0} = \frac{\alpha^2}{\cos^2\theta_0} = \alpha^2(1+\tan^2\theta_0) = \alpha^2 + \omega^2. \tag{h-8}$$

This may be rewritten as

$$\omega^2 = {\omega_0}^2 - \alpha^2 = \frac{1}{LC} - \left(\frac{R}{2L}\right)^2 \quad \text{with} \quad \omega_0 = \frac{1}{\sqrt{LC}} = 9.7 \times 10^2\ \mathrm{rad/s}, \tag{h-9}*$$

and we obtain

$$\omega = \sqrt{{\omega_0}^2 - \alpha^2} = 9.6428 \times 10^2\ \mathrm{rad/s} \cong 9.6 \times 10^2\ \mathrm{rad/s}. \tag{h-10}*$$

(i) From Eq. (h-2), the maximum value of $|I(t)|$ appears at $dI/dt = 0$ when the time is

$$t_{\mathrm{m}} = \frac{\theta_0}{\omega}. \tag{i-1}$$

From Eq. (2), the maximum value of $|I(t)|$ is then given by

$$I_{\mathrm{m}} = |I(t_{\mathrm{m}})| = \left(\frac{\alpha}{\cos\theta_0}\right) C V_0 e^{-\frac{\alpha}{\omega}\theta_0}. \tag{i-2}*$$

From Eqs. (3), (h-7) and (h-10), we have

$$\tan\theta_0 = \frac{\omega}{\alpha} = 10.568, \qquad \theta_0 = 1.4764\ \mathrm{rad}, \qquad t_{\mathrm{m}} = \frac{\theta_0}{\omega} = 1.531 \times 10^{-3}\ \mathrm{s}. \tag{i-3}$$

If I_{m} does not exceed I_{b} found in Problem (d), we must have

$$I_{\mathrm{m}} = |I(t_{\mathrm{m}})| \le I_{\mathrm{b}} \ \text{ or } \ \left(\frac{\alpha}{\cos\theta_0}\right) C\, V_0 e^{-\frac{\alpha}{\omega}\theta_0} \le I_{\mathrm{b}}, \tag{i-4}$$

which implies that the maximum value $V_{0\mathrm{b}}$ of V_0 occurs when the equality holds and is given by

$$V_{0\mathrm{b}} = \frac{I_{\mathrm{b}}}{\alpha C} e^{\frac{\alpha}{\omega}\theta_0} \cos\theta_0 = 2.0623 \times 10^3\ \mathrm{V} \cong 2.1 \times 10^3\ \mathrm{V}. \tag{i-5}*$$

[j] When $|I(t)|$ reaches its maximum at $t = t_\mathrm{m} = \theta_0/\omega$, the voltage of the capacitor has dropped from the initial voltage $V_0 = V_{0\mathrm{b}}$ to

$$V(t_\mathrm{m}) = \frac{Q(t_\mathrm{m})}{C} = \frac{V_{0\mathrm{b}}}{\sin\theta_0} e^{-\frac{\alpha}{\omega}\theta_0} \sin(2\theta_0) = 2V_{0\mathrm{b}} e^{-\frac{\alpha}{\omega}\theta_0} \cos\theta_0 \,. \qquad \text{(j-1)}$$

From $t = 0$ to $t = t_\mathrm{m}$, the energy supplied by the capacitor bank to the circuit, in the form of Joule heat and magnetic energy in the field of the coil, is

$$E_C = \frac{1}{2} C\{{V_{0\mathrm{b}}}^2 - [V(t_\mathrm{m})]^2\} = \frac{1}{2} C{V_{0\mathrm{b}}}^2 \left\{1 - 4e^{-\frac{2\alpha}{\omega}\theta_0} \cos^2\theta_0\right\}. \qquad \text{(j-2)}$$

By the law of conservation of energy, this entire amount of energy is eventually turned into heat in the coil and we have

$$\Delta E = E_C = \frac{1}{2} C{V_{0\mathrm{b}}}^2 \left\{1 - 4e^{-\frac{2\alpha}{\omega}\theta_0} \cos^2\theta_0\right\} = 2.0694 \times 10^4 \text{ J} \cong 2.1 \times 10^4 \text{ J}. \qquad \text{(j-3)*}$$

If the heat capacity (as computed in Eq. (f-3) remains about the same as that at $T = 293$ K, then the temperature increase ΔT is

$$\Delta T = \frac{\Delta E}{M c_p} = \frac{\Delta E}{\rho_m (\ell \pi b D) c_p} = 53 \text{ K}. \qquad \text{(j-4)*}$$

With such a temperature increase, the thermal and electrical properties of a metal such as copper do not change substantially.

Theoretical Question 2: Strong Resistive Electromagnets

MARKING SCHEME

Total	Mark(s)	Marking Scheme for Answers
Part A 1.4	(a) 1.0	x-component $B(x)$ of magnetic field (in terms of a, D, I, ℓ, μ_0). ➢ 0.2 for magnetic field $d\vec{B}$ from a current loop. →(a-1)† ➢ 0.2 for expressing $B(x)$ as a definite integral. →(a-2) ➢ 0.2 for carrying out the integration. →(a-2) ➢ 0.3 for expression of $B(x)$. →(a-2) ➢ 0.1 for sign of $B(x)$ or direction of magnetic field.
	(b) 0.4	Current I_0 (in terms of $a, D, B(0), \ell, \mu_0$) to make $B(0) = 10.0$ T. ➢ 0.2 for expression of I_0. →(b-2) ➢ 0.1 for significant figure (or mantissa) with first 2 digits correct. (b-2) ➢ 0.1 for unit and exponent.
Part B 3.0	(c) 1.2	Outward normal force per unit length $\Delta F_n/\Delta s$ (in terms of a, D', I, μ_0). ➢ 0.6 for $\bar{B} = \frac{1}{2}B(0)$. →(c-1) ➢ 0.2 for $B(0)$(when $\ell \to \infty$) $= \mu_0 I/a$. →(b-1) ➢ 0.2 for normal force on wire $\Delta F_n = I\bar{B}\Delta s$. →(c-2) ➢ 0.2 for expression of $\Delta F_n / \Delta s$. →(c-2)
	0.6	Tension F_t along the wire (in terms of a, D', I, μ_0). ➢ 0.2 for resultant of tension forces $-F_t\,\Delta\theta$. →(c-3) ➢ 0.2 for equilibrium condition $\Delta F_n = F_t(2\Delta s/D')$. →(c-4) ➢ 0.2 for expression of tension F_t. →(c-4)
	(d) 0.8	Current I_b at breaking of the turn (in terms of a, b, D, σ_b , μ_0). ➢ 0.2 for tensile stress $F_t/(ab) = \sigma_b$. →(d-1) ➢ 0.2 for $D' = 1.60\,D$. →(d-2) ➢ 0.2 for expression of I_b. →(d-3) ➢ 0.1 for significant figure (or mantissa) with first 2 digits correct. (d-3) 0.1 for unit and exponent.
	0.4	Magnetic field B_b at O when the current is I_b. ➢ 0.2 for expression of B_b. →(d-4) ➢ 0.1 for significant figure (or mantissa) with first 2 digits correct. (d-4) 0.1 for unit and exponent.

Part C 1.0	(e) 0.5	Power density of heat generation in the coil. ➢ 0.1 for expression of current density J in the wire, →(e-1) or volume τ of the wire. →(e-3) ➢ 0.2 for expression of power density. →(e-2) or (e-7) ➢ 0.1 for significant figure (or mantissa) with first 2 digits correct. (e-6) ➢ 0.1 for unit and exponent.
	(f) 0.5	Time rate of change $\dot{T}$ of temperature in the coil. ➢ 0.1 for heat capacity per unit volume $\rho_m c_p$, →(f-1) or heat capacity Mc_p. →(f-3) ➢ 0.2 for expression of $\dot{T}$. →(f-2) or (f-4) ➢ 0.1 for significant figure (or mantissa) with first 2 digits correct. (f-2) 0.1 for unit and exponent.
Part D 4.6	(g) 0.6	Expressions for the inductance L and resistance R of the coil. ➢ 0.1 for flux ϕ_B of a single turn. →(g-1) ➢ 0.1 for definition of $L = N\phi_B/I$. →(g-2) ➢ 0.2 for expression of L. →(g-2) ➢ 0.2 for expression of R. →(g-3)
	0.4	Values of the inductance L and resistance R of the coil. →(g-2) and (g-3) ➢ 0.1*2 for significant figure (or mantissa) with first 2 digits correct. ➢ 0.1*2 for unit and exponent.
	(h) 0.8	Expressions for α and ω (in terms of R, L, and C). ➢ 0.1 for loop equation for voltage. →(h-1) ➢ 0.1 for conditions on α and ω. →(h-4), (h-5) ➢ 0.3 for expression of α. →(h-7) ➢ 0.3 for expression of ω. →(h-9)
	0.4	Values of α and ω. →(h-7) and (h-10) ➢ 0.1*2 for significant figure (or mantissa) with first 2 digits correct. ➢ 0.1*2 for unit and exponent.
	(i) 0.6	Expression for I_{m} (in terms of α, ω, θ_0, V_0 and C). ➢ 0.1 for condition of maximum current $dI/dt = 0$. ➢ 0.3 for obtaining the instant t_{m} of maximum current. →(i-1) ➢ 0.2 for expression of I_{m}. →(i-2)
	0.4	Maximum value $V_{0\mathrm{b}}$ for which I_{m} will not exceed I_{b} of Problem (d). ➢ 0.3 for significant figure (or mantissa) with first 2 digits correct.→(i-5) ➢ 0.1 for unit and exponent.

	(j) 1.0	The total amount of heat ΔE dissipated in the coil. ➢ 0.3 for energy E_C supplied by the capacitor up to the time t_m →(j-2) ➢ 0.2 for equality between E_C and the amount of heat ΔE. →(j-2) ➢ 0.3 for expression of total amount of heat ΔE. →(j-3) ➢ 0.1 for significant figure (or mantissa) with first 2 digits correct.→(j-3) ➢ 0.1 for unit and exponent.
	0.4	The temperature increase ΔT of the coil. ➢ 0.1 for expression of ΔT. →(j-4) ➢ 0.2 for significant figure (or mantissa) with first 2 digits correct.→(j-4) ➢ 0.1 for unit and exponent.

†The equation number(s) at the end of a line refers to equation(s) in the SOLUTION sheets.

Theoretical Question 3: Electron and Gas Bubbles in Liquids

SOLUTION

Part A. An Electron Bubble in Liquid Helium

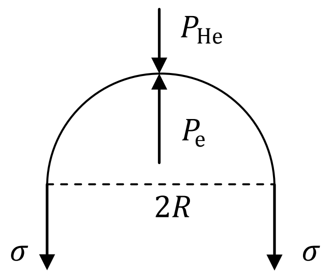
(a) Consider a half of the spherical interface (see Fig. A1 below). The condition for its static equilibrium implies that the total force acting on it must be zero. This implies

$$\pi R^2 (P_e - P_{He}) = 2\pi R\sigma \tag{a-1}$$

which leads to

$$P_e = P_{He} + \frac{2\sigma}{R} \tag{a-2}^*$$

Fig. A1



**An equation marked with an asterisk gives key answers to the problem.*

According to the de Broglie relation $p = h/\lambda \propto 1/R$, the non-relativistic kinetic energy E_k is inversely proportional to R^2, i.e.

$$E_k = \frac{p^2}{2m} = \frac{\text{const.}}{R^2}. \tag{a-3}$$

By the work- energy theorem, we have

$$-P_e dV = dE_k = (-2)\frac{\text{const.}}{R^3}dR = -\frac{2}{R}E_k dR \tag{a-4}$$

Thus,

$$-P_e(4\pi R^2 dR) = -\frac{2}{R}E_k\, dR \tag{a-5}$$

or

$$P_e = \frac{1}{2\pi R^3}E_k \tag{a-6}^*$$

[Alternative]

The state of an electron confined in the bubble corresponds to standing waves which vanish on the interface. According to Part B of Question 1, these are equivalent to the superposition of two travelling waves moving in opposite directions and continually being reflected at the interface. They give rise to pressure on the interface and the relation between the non-relativistic kinetic energy E_{k} and the pressure P_{e} for an electron inside a bubble is similar to that obtained from the kinetic theory of gases. Thus we have

$$P_{\mathrm{e}}=\frac{2}{3}\frac{E_{\mathrm{k}}}{V}=\frac{2}{4\pi R^3}E_{\mathrm{k}}=\frac{1}{2\pi R^3}E_{\mathrm{k}} \qquad \text{(a-7)*}$$

(b) Let $\hbar=h/(2\pi)$. From the uncertainty relations, we have

$$\Delta x\,\Delta p_x \geq \frac{1}{2}\hbar,\qquad \Delta y\,\Delta p_y \geq \frac{1}{2}\hbar,\qquad \Delta z\,\Delta p_z \geq \frac{1}{2}\hbar. \qquad \text{(b-1)}$$

From symmetry considerations implied by isotropy, we have

$$\bar{x}=\bar{y}=\bar{z}=0,\quad \overline{p_x}=\overline{p_y}=\overline{p_z}=0 \qquad \text{(b-2)}$$

$$(\Delta x)^2=\overline{x^2}-\bar{x}^2=\overline{x^2}=(\Delta y)^2=(\Delta z)^2,\quad (\Delta p_x)^2=\overline{p_x{}^2}=(\Delta p_y)^2=(\Delta p_z)^2. \qquad \text{(b-3)}$$

where $\bar{f}$ denotes the mean value of the quantity f. Therefore, we have

$$3(\Delta \mathrm{x})^2=(\Delta \mathrm{x})^2+(\Delta \mathrm{y})^2+(\Delta \mathrm{z})^2=\overline{\mathrm{x}^2}+\overline{\mathrm{y}^2}+\overline{\mathrm{z}^2}=\overline{\mathrm{r}^2}. \qquad \text{(b-4)}$$

$$3(\Delta p_x)^2=(\Delta p_x)^2+(\Delta p_y)^2+(\Delta p_z)^2=\overline{p_x^2}+\overline{p_y^2}+\overline{p_z^2}=\overline{p^2}. \qquad \text{(b-5)}$$

Thus we obtain (cf. 18th IPhO)

$$\overline{r^2}\,\overline{p^2}=9(\Delta x)^2(\Delta p_x)^2\geq\frac{9}{4}\hbar^2 \qquad \text{(b-6)}$$

and the kinetic energy must satisfy the following inequality:

$$E_{\mathrm{k}}=\frac{\overline{p^2}}{2m}\geq\frac{1}{2m}\left(\frac{9\hbar^2}{4}\right)\frac{1}{\overline{r^2}} \qquad \text{(b-7)}$$

The smallest possible kinetic energy E_0 of the electron consistent with the uncertainty relations is thus obtained if the mean-squared-radius $\overline{r^2}$ is set equal to its largest possible value of R^2. This gives

$$E_{\mathrm{k}}\geq E_0=\frac{9\hbar^2}{8mR^2} \qquad \text{(b-8)*}$$

(c) If $E_{\mathrm{k}}=E_0$, it follows from Eqs. (a-2), (a-3), and (b-7) that we have

$$P_{\mathrm{e}}=\frac{1}{2\pi R^3}E_0=\frac{9\hbar^2}{16m\pi R^5}=\frac{2\sigma}{R}+P_{\mathrm{He}} \qquad \text{(c-1)}$$

For $P_{\mathrm{He}}=0$, this gives the following equilibrium radius of the electron bubble:

$$R_{\mathrm{e}}=\left(\frac{9\hbar^2}{32\pi m\sigma}\right)^{\frac{1}{4}}=\left(\frac{9\times(1.055\times10^{-34})^2}{32\pi\times9.11\times10^{-31}\times3.75\times10^{-4}}\right)^{\frac{1}{4}}$$

$$=(2.91674\times10^{-36})^{1/4}\ \mathrm{m}\cong1.31\ \mathrm{nm} \qquad \text{(c-2)*}$$

It might be of some interest to note that, from Eq. (c-1), the corresponding minimum kinetic energy is

$$E_0 = 4\pi R_0^2\sigma = 3h\left(\frac{\sigma}{2\pi m}\right)^{1/2} = 0.100\text{ eV} \tag{c-3}$$

(d) The condition for stable local equilibrium of the electron bubble at radius R is that when R is increased by a small amount $dR > 0$, the inward force pushing on the interface must be greater than the outward force so as to decrease the radius. Thus, from Eq. (c-1), we obtain

$$\frac{2\sigma}{(R+dR)} + P_{\text{He}} > \frac{9\hbar^2}{16m\pi(R+dR)^5} \tag{d-1}$$

By keeping only terms linear in dR after both sides of the inequality are expanded as a power series and making use of Eq. (c-1) to eliminate P_{He}, we obtain

$$(-1)\frac{2\sigma}{R^2} > (-5)\frac{9\hbar^2}{16m\pi R^6} \tag{d-2}$$

Note that the same inequality is obtained if we consider a small change $dR < 0$. Using Eq. (c-1), we may express Eq. (d-2) in terms of P_{He} as

$$\frac{2\sigma}{R} < 5\left(P_{\text{He}} + \frac{2\sigma}{R}\right) \tag{d-3}$$

or equivalently,

$$P_{\text{He}} > -\left(\frac{8\sigma}{5R}\right) \tag{d-4)*}$$

(e) From Eqs. (a-2), (a-6), and (b-8), we have

$$\frac{2\sigma}{R} + P_{\text{He}} = P_e = \frac{E_k}{2\pi R^3} \geq \frac{E_0}{2\pi R^3} = \frac{9\hbar^2}{16m\pi R^5} \tag{e-1}$$

or equivalently,

$$P_{\text{He}} \geq \frac{9\hbar^2}{16m\pi R^5} - \frac{2\sigma}{R} \tag{e-2}$$

The minimum of the right-hand side of the inequality occurs when its derivative vanishes, i.e.

$$\frac{-45\hbar^2}{16m\pi R^6} + \frac{2\sigma}{R^2} = 0 \tag{e-3}$$

or

$$R = R_{\text{th}} = \left(\frac{45\hbar^2}{32m\pi\sigma}\right)^{1/4} \tag{e-4}$$

Substituting the last result back into Eq. (e-2), we obtain

$$P_{\text{He}} \geq P_{\text{th}} \equiv \frac{9\hbar^2}{16m\pi R_{\text{th}}^5} - \frac{2\sigma}{R_{\text{th}}} = \left(\frac{1}{5} - 1\right)\frac{2\sigma}{R_{\text{th}}} = \frac{-8\sigma}{5R_{\text{th}}} = \frac{-16\sigma}{5}\left(\frac{2m\pi\sigma}{45\hbar^2}\right)^{1/4} \tag{e-5)*}$$

For $P_{He} < P_{th}$, no equilibrium is possible for the electron bubble.

Part B. Single Gas Bubble in Liquid — Collapsing and Radiation

(f) When the bubble's radius R changes by dR, the volume of the liquid displaced by the interface is $dV = 4\pi R^2 dR$. But the total volume of the incompressible liquid cannot change, so the change of the volume at the outer surface of the liquid must also be dV. Thus the amount of work done on the liquid is

$$dW = PdV - P_0 dV = (P - P_0)4\pi R^2 dR \qquad \text{(f-1)*}$$

From Eq.(2), the change in total kinetic energy of the liquid is, in the limit $r_0 \to \infty$,

$$dE_\mathrm{k} = d\left[2\pi\rho_0 R^4 \dot{R}^2\left(\frac{1}{R} - \frac{1}{r_0}\right)\right] = 2\pi\rho_0 d\left(R^3\dot{R}^2\right). \qquad \text{(f-2)}$$

Since $dE_\mathrm{k} = dW$, we obtain

$$\frac{1}{2}\rho_0 d\left(R^\mathrm{m}\dot{R}^2\right) = (P - P_0)R^\mathrm{n} dR. \qquad \text{(f-3)}$$

with

$$\mathrm{m} = 3, \qquad \mathrm{n} = 2. \qquad \text{(f-4)*}$$

(g) The initial gas temperature is T_0. According to the ideal gas law, the initial gas pressure $P_\mathrm{i} = P(R_\mathrm{i})$ is thus given by

$$P_\mathrm{i} R_\mathrm{i}^3 = P_0 R_0^3. \qquad \text{(g-1)*}$$

Since the process is adiabatic, the radial dependence of the gas pressure P is

$$P \equiv P(R) = \left(\frac{R_\mathrm{i}^3}{R^3}\right)^\gamma P_\mathrm{i} = \left(\frac{R_\mathrm{i}}{R}\right)^5 P_\mathrm{i} = \left(\frac{R_\mathrm{i}}{R}\right)^5 P_0 \left(\frac{R_0}{R_\mathrm{i}}\right)^3. \qquad \text{(g-2)*}$$

and the temperature T corresponding to the radius R is given by

$$T \equiv T(R) = \left(\frac{R_\mathrm{i}^3}{R^3}\right)^{(\gamma-1)} T_0 = \left(\frac{R_\mathrm{i}}{R}\right)^2 T_0. \qquad \text{(g-3)*}$$

(h) From Eqs. (3) and (g-2), we have

$$\frac{1}{2R^2}\frac{d}{dR}\left(R^3\dot{R}^2\right) = \frac{P - P_0}{\rho_0} = \frac{P_0}{\rho_0}\left[\frac{P_i}{P_0}\left(\frac{R_i}{R}\right)^{3\gamma} - 1\right] \qquad \text{(h-1)}$$

In terms of $\beta = R/R_\mathrm{i}$ and $\dot{\beta} = \dot{R}/R_\mathrm{i}$, the last equation may be rewritten as

$$\frac{1}{2\beta^2}\frac{d}{d\beta}\left(\beta^3\dot{\beta}^2\right) = \frac{P_0}{\rho_0 R_\mathrm{i}^2}\left(\frac{P_\mathrm{i}}{P_0}\beta^{-3\gamma} - 1\right). \qquad \text{(h-2)}$$

This may be integrated to give

$$\frac{1}{2}\beta^3\dot{\beta}^2 = \frac{P_0}{\rho_0}\int_1^\beta \left(\frac{P_\mathrm{i}}{P_0}y^{2-3\gamma} - y^2\right)dy = \frac{P_0}{\rho_0}\left[\left(\frac{P_\mathrm{i}}{P_0}\right)\frac{\beta^{3-3\gamma} - 1}{3(1-\gamma)} - \frac{\beta^3 - 1}{3}\right]$$

$$= \frac{P_0}{3\rho_0\beta^2}\left[-\left(\frac{P_i}{P_0}\right)\frac{1}{(\gamma-1)}(1-\beta^2)+\beta^2(1-\beta^3)\right]. \quad \text{(h-3)}$$

Since $Q \equiv P_i/[(\gamma-1)P_0]$ and $\gamma = 5/3$, the last equation leads to

$$\frac{1}{2}\rho_0\dot{\beta}^2 = -U(\beta) \equiv \frac{-P_0}{3R_i^2\beta^5}[Q(1-\beta^2)-\beta^2(1-\beta^3)]$$

$$= \frac{-P_0(1-\beta^2)}{3R_i^2\beta^5}\left[Q-\frac{\beta^2(1-\beta^3)}{(1-\beta^2)}\right]. \quad \text{(h-4)}$$

Thus we obtain

$$\mu = \frac{P_0}{3R_i^2}. \quad \text{(h-5)*}$$

(i) The radius of the bubble reaches its minimum value when $\dot{R} = R_i\dot{\beta} = 0$. Thus, from Eq. (h-4), we obtain

$$Q = \frac{\beta_m^2}{1-\beta_m^2}(1-\beta_m^3) = \beta_m^2\left(1+\frac{\beta_m^2}{1+\beta_m}\right). \quad \text{(i-1)}$$

The last equality shows that β_m must be very small in order that $Q \ll 1$. Thus

$$Q \approx \beta_m^2, \text{ or } \beta_m \approx \sqrt{Q}, \text{ i.e. } C_m = 1 \quad \text{(i-2)*}$$

For $R_i = 7R_0 = 35.0\ \mu\text{m}$, we have, from Eq. (g-1),

$$Q = \frac{P_i}{P_0(\gamma-1)} = \frac{1}{(\gamma-1)}\left(\frac{R_0}{R_i}\right)^3 = \frac{3}{2}\left(\frac{1}{7}\right)^3 = 0.00437. \quad \text{(i-3)}$$

Therefore

$$\beta_m = \sqrt{Q} = 0.0661, \quad \text{(i-4)}$$

$$R_m = \beta_m R_i = \sqrt{Q}R_i = 0.0661 \times 35\ \mu\text{m} = 2.31\ \mu\text{m}, \quad \text{(i-5)*}$$

and from Eq. (g-3), the corresponding temperature T_m is

$$T_m = \left(\frac{1}{\beta_m}\right)^2 T_0 = \left(\frac{1}{0.0661}\right)^2 \times 300\text{ K} = 6.86\times 10^4\text{ K}. \quad \text{(i-6)*}$$

(j) From Eq. (h-4), the maximum value of the radial speed $u \equiv |\dot{\beta}|$ occurs at $\beta = \beta_u$ where $-U(\beta)$ is also at its maximum, i.e. the derivative of $U(\beta)$ with respect to β must vanish at $\beta = \beta_u$. Since

$$U(\beta) = \frac{P_0}{3R_i^2}\left[Q\left(\frac{1}{\beta^5}-\frac{1}{\beta^3}\right)-\left(\frac{1}{\beta^3}-1\right)\right], \quad \text{(j-1)}$$

we have

$$\left.\frac{dU}{d\beta}\right|_{\beta=\beta_u} = \frac{-P_0}{3R_i^2\beta_u}\left[Q\left(\frac{5}{\beta_u^5}-\frac{3}{\beta_u^3}\right)-\frac{3}{\beta_u^3}\right] = 0. \quad \text{(j-2)}$$

Thus

$$Q = \frac{3\beta_u^2}{(5-3\beta_u^2)}, \text{ or } \beta_u^2 = \frac{5}{3}\left(\frac{Q}{1+Q}\right). \quad \text{(j-3)}$$

which implies

$$\beta_u = \sqrt{\frac{5}{3}\left(\frac{Q}{1+Q}\right)} = 0.0852. \tag{j-4)*$$

The radius midway between β_u (corresponding to maximum speed) and $\beta_{\rm m}$ (corresponding to zero speed) is given by

$$\bar{\beta} \equiv \frac{1}{2}(\beta_{\rm m} + \beta_u) \cong \frac{1}{2}(0.0661 + 0.0852) = 0.0757 \qquad \text{(j-5)}$$

From Eq. (h-4), the dimensionless radial speed at radius $\bar{\beta}$ is

$$\bar{u} = -\dot{\beta}(\bar{\beta}) = \sqrt{\frac{-2}{\rho_0}U(\bar{\beta})}$$

$$= \sqrt{\frac{2P_0(1-\bar{\beta}^2)}{3\rho_0 R_{\rm i}^2 \bar{\beta}^3}\left[1 - \frac{Q}{\bar{\beta}^2} + \frac{\bar{\beta}^2}{(1+\bar{\beta})}\right]} = 5.52 \times 10^6. \qquad \text{(j-6)*}$$

Thus an estimate of the duration $\Delta t_{\rm m}$ for the radius of the bubble to diminish from β_u to the minimum value $\beta_{\rm m}$ is

$$\Delta t_{\rm m} = \frac{(\beta_u - \beta_{\rm m})}{\bar{u}} = \frac{(0.0852 - 0.0661)}{5.52 \times 10^6} = 3.45 \times 10^{-9}\ \text{s}. \qquad \text{(j-7)*}$$

(k) Suppose the bubble is a surface radiator with emissivity a. By making use of Eq. (g-3), the radiant power $W_{\rm r}$ of the bubble at temperature T can be written as a function of β, i.e.

$$W_{\rm r} = a(\sigma_{\rm SB}T^4)4\pi R^2 = 4\pi R_{\rm i}^2 a\sigma_{\rm SB} T_0^4 \frac{1}{\beta^6}, \qquad \text{(k-1)}$$

where $\sigma_{\rm SB}$ is the Stefan-Boltzmann constant. The power supplied to the bubble is

$$\dot{E} = -P\frac{dV}{dt} = -P_{\rm i}\left(\frac{V_{\rm i}}{V}\right)^{\gamma}\frac{dV}{dt} = -4\pi R_{\rm i}^3 P_{\rm i}\frac{\dot{\beta}}{\beta^3} \qquad \text{(k-2)*}$$

The assumption of an adiabatic collapsing of the bubble is deemed reasonable when the radiant power is less than 20 % of the power supplied to the bubble at $\beta = \bar{\beta}$. Thus we have

$$4\pi R_{\rm i}^2 a\sigma_{\rm SB} T_0^4 \frac{1}{\bar{\beta}^6} \leq 4\pi R_{\rm i}^3 P_{\rm i}\frac{\bar{u}}{\bar{\beta}^3} \times 20\ \% \qquad \text{(k-3)}$$

or

$$a \leq \frac{P_{\rm i} R_{\rm i}}{5\sigma_{\rm SB} T_0^4}\bar{\beta}^3\bar{u} = \frac{P_0 R_{\rm i}}{5\sigma_{\rm SB} T_0^4}\left(\frac{R_0}{R_{\rm i}}\right)^3 \bar{\beta}^3\bar{u} = 0.0107 \qquad \text{(k-4)*}$$

Theoretical Question 3: Electron and Gas Bubbles in Liquids

MARKING SCHEME

Total	Mark(s)	Marking Scheme for Answers
Part A 4.0	(a) 0.4	Relation between P_{He}, P_{e}, and σ. ➢ 0.1 for force on interface from surface tension. →(a-1)† ➢ 0.1 for force on interface from pressure. →(a-1) ➢ 0.1 for condition of static equilibrium. →(a-1) ➢ 0.1 for $P_{\mathrm{e}} = P_{\mathrm{He}} + 2\sigma/R$ →(a-2)
	1.0	Relation between E_k and P_{e}. ➢ 0.1 for $P \propto 1/R$ ➢ 0.2 for $E_{\mathrm{k}} \propto 1/R^2$ →(a-3) ➢ 0.2 for $dE_{\mathrm{k}}/dR \propto -2E_{\mathrm{k}}/R$ →(a-4) ➢ 0.2 for work-energy relation $dE_{\mathrm{k}} = dW$ →(a-4) ➢ 0.1 for work supplied to bubble $dW = -P_{\mathrm{e}}dV$ →(a-5) ➢ 0.2 for $P_{\mathrm{e}} = E_{\mathrm{k}}/2\pi R^3$ →(a-6)
	(b) 0.8	The smallest possible kinetic energy E_0 as a function of R. ➢ 0.2 for uncertainties = mean-squared values →(b-2)(b-3) ➢ 0.1 for $(\Delta x)^2 = (\Delta y)^2 = (\Delta z)^2$ →(b-4)(b-5) ➢ 0.1 for $3(\Delta x)^2 = \overline{r^2}$ →(b-4)(b-5) ➢ 0.2 for $\overline{r^2}\,\overline{p^2} \geq 9\hbar^2/4$ →(b-6) ➢ 0.2 for $E_0 = 9\hbar^2/8mR^2$ →(b-8)
	(c) 0.6	The bubble's equilibrium radius R_{e} when $E_{\mathrm{k}} = E_0$ and $P_{\mathrm{He}} = 0$. ➢ 0.2 for establishing a valid equation for R_{e}. →(c-1) ➢ 0.2 for $R_{\mathrm{e}} = (9\hbar^2/32\pi m\sigma)^{1/4}$ →(c-2) ➢ 0.1 for significant figure (or mantissa) with first 2 digits correct. ➢ 0.1 for unit and exponent
	(d) 0.6	Condition satisfied by R and P_{He} for stable equilibrium at R. ➢ 0.2 for indicating stable equilibrium requires restoring force. →(d-1) ➢ 0.4 for $P_{\mathrm{He}} > -8\sigma/(5R)$ →(d-4)
	(e) 0.6	The threshold pressure P_{th} for no possibility of equilibrium. ➢ 0.2 for an inequality relating P_{He} and R. →(e-1) ➢ 0.4 for expression of P_{th}. →(e-5)

Part B $\boxed{6.0}$	(f) 0.4	Amount of work dW done on the liquid from R to $R+dR$. ➢ 0.1 for work on a sphere is $dW = FdR = PdV$ →(f-1) ➢ 0.1 for change of liquid volume is zero. →(f-1) ➢ 0.2 for expression of dW →(f-1)
	0.4	Values of m and n. ➢ 0.2 for $\mathrm{m}=3$ →(f-4) ➢ 0.2 for $\mathrm{n}=2$ →(f-4)
	(g) 0.4	Pressure $P \equiv P(R)$ as a function of R. ➢ 0.2 for obtaining initial pressure P_{i} →(g-1) ➢ 0.1 for $PV^{\gamma} =$ constant →(g-2) ➢ 0.1 for expression of $P(R)$ →(g-2)
	0.2	Temperature $T \equiv T(R)$ as a function of R. ➢ 0.1 for $TV^{\gamma-1} =$ constant →(g-3) ➢ 0.1 for expression of $T(R)$ →(g-3)
	(h) 0.6	The coefficient μ in terms of R_{i} and P_0. ➢ 0.2 for an integrable equation for $\dot{\beta}^2$ →(h-2) ➢ 0.2 for carrying out integration leading to $U(\beta)$ →(h-3) ➢ 0.2 for $\mu = P_0/(3R_{\mathrm{i}}^2)$ →(h-5)
	(i) 0.4	Values of the constant C_{m}. ➢ 0.4 for $C_{\mathrm{m}} = 1$ →(i-2)
	0.3	The minimum radius R_{m} for $R_{\mathrm{i}} = 7R_0$. ➢ 0.2 for significant figure (or mantissa) with first 2 digits correct. (i-5) ➢ 0.1 for unit and exponent
	0.3	The temperature T_{m} of the gas at the minimum radius $\beta = \beta_{\mathrm{m}}$. ➢ 0.2 for significant figure (or mantissa) with first 2 digits correct. (i-6) ➢ 0.1 for unit and exponent
	(j) 0.6	The radius β_u at which $u \equiv \lvert\dot{\beta}\rvert$ reaches its maximum value. ➢ 0.1 for an equation satisfied by β_u →(j-2) ➢ 0.2 for obtaining β_u as a function of Q →(j-4) ➢ 0.2 for significant figure (or mantissa) with first 2 digits correct. (j-4) ➢ 0.1 for unit and exponent
	0.4	The value $\bar{u}$ of radial speed u at $\beta = \bar{\beta} \equiv (\beta_m + \beta_u)/2$. ➢ 0.1 for value of $\beta = \bar{\beta} \equiv (\beta_m + \beta_u)/2$ ➢ 0.2 for significant figure (or mantissa) with first 2 digits correct. (j-6) ➢ 0.1 for unit and exponent

	0.6	The time duration Δt_m for β to diminish from β_u to β_m. ➢ 0.1 for a formula to compute Δt_m →(j-7) ➢ 0.1 for choosing a reasonable radial speed such as $\bar{u}$ in $\Delta t_m = (\beta_u - \beta_m)/\bar{u}$ →(j-7) ➢ 0.1 for the value of a reasonable radial speed. →(j-6) ➢ 0.2 for significant figure (or mantissa) with first 2 digits correct. (j-7) ➢ 0.1 for unit and exponent
	(k) 0.6	The power $\dot{E}$ supplied to the bubble at β. ➢ 0.2 for a formula of $\dot{E}$ in terms of derivatives $\dot{V}$ or $\dot{T}$. →(k-2) ➢ 0.2 for expressing $\dot{V}$ or $\dot{T}$ in terms of $\dot{\beta}$ and β. →(k-2) ➢ 0.2 for $\dot{E} = -4\pi R_i^3 P_i \dot{\beta}/\beta^3$ →(k-2)
	0.8	The upper bound of the emissivity a. ➢ 0.2 for radiant power in terms of temperature →(k-1) ➢ 0.2 for radiant power in terms of β →(k-1) ➢ 0.2 for an upper bound of a →(k-3) ➢ 0.2 for value of $a < 0.0107$ →(k-4)

†The equation number(s) at the end of a line refers to equation(s) in the SOLUTION sheets.