

Experimental Question 1: Levitation of Conductors in an Oscillating Magnetic Field

SOLUTION

a. Using Faraday's law:

$$\epsilon(t) = -\frac{d\Phi_B}{dt} = -\sqrt{2}\omega\Phi_B^{rms} \cos(\omega t)$$

The overall sign will not be graded.

For the current, we use the extensive hints in the question to write:

$$I(t) = -\frac{\sqrt{2}\omega\Phi_B^{rms}}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \delta) = -\frac{\sqrt{2}\omega\Phi_B^{rms}}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1} \frac{\omega L}{R}\right)$$

This can be rewritten in a way which will be useful for part (c):

$$I(t) = -\frac{\sqrt{2}\omega\Phi_B^{rms}}{\sqrt{R^2 + \omega^2 L^2}} (\cos \delta \cos(\omega t) + \sin \delta \sin(\omega t)) = -\frac{\sqrt{2}\omega\Phi_B^{rms}}{R^2 + \omega^2 L^2} (R \cos(\omega t) + \omega L \sin(\omega t))$$

In the last equality, we used $\tan \delta = \omega L/R$ to derive $\sin \delta = \omega L/\sqrt{R^2 + \omega^2 L^2}$ and $\cos \delta = R/\sqrt{R^2 + \omega^2 L^2}$.

All the above forms of the answer will be accepted. The overall sign will not be graded.

b. Let us forget about the metal ring, and consider a cylindrical surface at some distance z from the solenoid, with radius r and an infinitesimal height dz . The magnetic Gauss law implies that the flux through the cylinder's wall should cancel the net flux through its bases:

$$2\pi r dz B_r + \Phi_B(z + dz) - \Phi_B(z) = 0$$

Where Φ_B is the flux of the vertical magnetic field through each circular base. Dividing by dz and doing an infinitesimal amount of algebra, we get:

$$B_r = -\frac{1}{2\pi r} \cdot \frac{d\Phi_B}{dz}$$

c. The field oscillations are very slow with respect to the transit time of light through the system. Therefore, Φ_B oscillates with the same phase at all heights z , and we have:

$$B_r = -\frac{1}{\sqrt{2}\pi r} \cdot \frac{d\Phi_B^{rms}}{dz} \sin(\omega t)$$

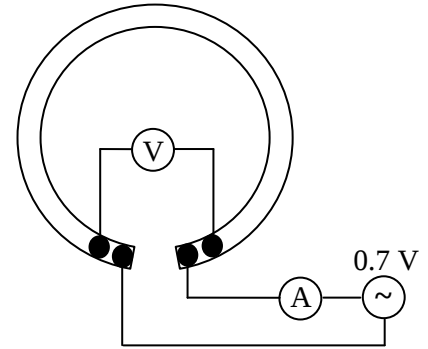
Then the momentary force reads:

$$F(t) = -2\pi r I(t) B_r(t) = -\frac{2\omega\Phi_B^{rms}}{R^2 + \omega^2 L^2} \cdot \frac{d\Phi_B^{rms}}{dz} \sin(\omega t) (R \cos(\omega t) + \omega L \sin(\omega t))$$

The time-average of $\sin(\omega t) \cos(\omega t)$ is zero, while the average of $(\sin(\omega t))^2$ is $1/2$. Therefore, the time-averaged force reads:

$$\langle F \rangle = -\frac{\omega^2 L \Phi_B^{rms}}{R^2 + \omega^2 L^2} \cdot \frac{d\Phi_B^{rms}}{dz} = -\frac{L \epsilon^{rms}}{R^2 + \omega^2 L^2} \cdot \frac{d\epsilon^{rms}}{dz} = -\frac{L}{2(R^2 + \omega^2 L^2)} \cdot \frac{d(\epsilon^{rms})^2}{dz}$$

d. The ring's resistance is much smaller than the resistance of the electric wires and their contacts. If the voltmeter and ammeter are connected to the same two points on the ring, the measured resistance would be on the order of 0.1Ω , which is almost entirely due to the contacts. Using the multimeters on ohm-meter mode is also pointless for such resistances. Therefore, a four-terminal circuit is necessary, as shown in the figure. The contact with the ring is accomplished by snapping the "crocodiles" onto it. Note that the voltmeter's contacts must be to the inside from the current's contacts. Since no resistor is used, the power supply is effectively short-circuited, with the total resistance (and therefore, the current) mostly determined by the wires and contacts. With an optimal use of wires, a current of over 6A can be obtained. This results in a voltage of about 10mV on the ring. The ammeter's accuracy is 0.01A, while the voltmeter's accuracy is 0.1mV. This makes the voltage the primary source of measurement error, whose relative value is 1%. During the circuit's operation, the current and voltage on the ring steadily increase. Therefore, it is necessary to take the current and voltage readings simultaneously. To minimize and estimate the error in this procedure, 3 sets of measurements should be taken at slightly different values of the current (without changing the circuit). These measurements from a sample experiment are reproduced in Table 1.



$I \text{ (A)} \pm 0.01\text{A}$	$V \text{ (mV)} \pm 0.1\text{mV}$	$R_{raw} = V/I \text{ (m}\Omega) \pm 1\%$
6.9	11.3	1.638
6.34	10.4	1.640
6.6	10.8	1.636

Table 1: Sample measurements of current and voltage on the thin ring.

In this case, we see that the statistical fluctuations are much smaller than the measurement error. In other cases, they come out similar. A reasonable estimate for the error in R_{raw} would be between 0.5% and 1%. Choosing 0.5% in our case, we write $R_{raw} = 1.638\text{m}\Omega \pm 0.008\text{m}\Omega$ (0.5%).

The resistance R_{raw} is not the resistance of the entire ring, but only of the stretch between the two voltage terminals. To take this into account, we must know the distance d_{ter} between the terminals, the gap d_{gap} between the ring's ends and the average circumference πD_{thin} of the ring. The terminal distance in our sample experiment was $d_{ter} = 2.2\text{cm} \pm 0.2\text{cm}$, with the error due to the width of the contacts. The gap was $d_{gap} = 1.0\text{cm} \pm 0.05\text{cm}$, with the error due to the ruler's resolution and the width of the ring. For the arc angles associated with d_{ter} and d_{gap} , we may treat arcs as straight lines, with a negligible error of about 0.02cm.

The best way to find the average circumference is to measure the ring's outer and inner diameters with the ruler and take their average. The results are $D_{thin}^> = 9.60\text{cm} \pm 0.05\text{cm}$ and $D_{thin}^< = 9.00\text{cm} \pm 0.05\text{cm}$. Therefore, $D_{thin} = 9.30\text{cm} \pm 0.04\text{cm}$ (0.35%), with the errors due to the ruler's resolution. Equivalently, a measurement with the same

accuracy can be made by placing the ring on a sheet of millimeter paper. The average circumference is now $\pi D_{thin} = 29.2\text{cm} \pm 0.1\text{cm}$ (0.35%).

Other methods, such as measuring the average diameter by taking the maximal distance between an inner point and an outer point of the ring lead to an higher error of 0.5%. Taking the inner or outer diameter *instead* of the average diameter introduces an error of about 3mm, i.e. 3%.

The true resistance of the thin ring now reads:

$$R_{thin} = R_{raw} \frac{\pi D_{thin} - d_{gap}}{\pi D_{thin} - d_{ter}} = 1.711\text{m}\Omega \quad (1)$$

To estimate the error, we write:

$$R_{thin} \approx R_{raw} \left(1 + \frac{d_{ter} - d_{gap}}{\pi D_{thin}} \right)$$

The error of the quantity in parentheses is mainly due to $\Delta d_{ter}/(\pi D_{thin}) = 0.2\text{cm}/30\text{cm} = 0.007 = 0.7\%$. Combining this with the 0.5% error in R_{raw} , we get:

$$\frac{\Delta R_{thin}}{R_{thin}} = 0.85\%; \quad \Delta R_{thin} = 0.015 \text{ m}\Omega$$

The distribution of sample measurement results on several different rings is consistent with this error estimate.

Neglecting to take d_{gap} into account introduces an error of $d_{gap}/(\pi D_{thin}) = 3\%$. Neglecting to take d_{ter} into account introduces an error of $d_{ter}/(\pi D_{thin}) = 6\%$. Forgetting about both and just using R_{raw} introduces an error of $(d_{ter} - d_{gap})/(\pi D_{thin}) = 3\%$.

A slightly inferior alternative to using a small d_{ter} is to connect the voltage terminals at diametrically opposite points of the ring. This decreases the measured voltage by a factor of 2, increasing its relative error by the same factor. The error in R_{raw} then becomes about 1%, slightly increasing the final error in R_{thin} .

e. As can be seen from the rings' cross-sections, the resistance R of the closed ring is smaller than R_{thin} by an order of magnitude. This makes a naïve 2-terminal measurement even more hopeless. A 4-terminal measurement as in part (d) is possible, but will result in a large error of about 5% due to the voltmeter's resolution. Furthermore, for the closed ring the inductive impedance ωL is no longer negligible, and will introduce a systematic error of about 3%.

The optimal solution is to use the fact that the rings are made of the same material, and deduce R from the rings' geometries and the accurately measured R_{thin} :

$$R = R_{thin} \frac{\pi D_{closed}}{\pi D_{thin} - d_{gap}} \cdot \frac{A_{thin}}{A_{closed}}$$

where A stands for the cross-section area. The average diameter D_{closed} of the closed ring can be found as in part (d), with the results $D_{closed}^> = 10.05\text{cm} \pm 0.05\text{cm}$ and $D_{closed}^< = 7.60\text{cm} \pm 0.05\text{cm}$. Therefore, $D_{closed} = 8.82\text{cm} \pm$

0.03cm. In this case, using the inner or outer diameter instead of the average one introduces a large error of 1.2cm, i.e. over 10%.

Measuring A_{thin} directly also introduces large errors. Measuring the thin ring's thickness and height with the ruler introduces an error of $0.5\text{mm}/3\text{mm} = 15\%$ for each dimension. Multiplying by $\sqrt{2}$, this implies an error of 25% in the area. A student may also try to measure the ring's dimensions using the screw attached to the solenoid. Then the measurement error for each dimension decreases to about 1/16 of a screw step, i.e. $0.09\text{mm}/3\text{mm} = 3\%$, which implies a 4% error in the area.

The solution is to weigh the rings using the digital scale. We then have:

$$\frac{A_{thin}}{A_{closed}} = \frac{m_{thin}}{m_{closed}} \cdot \frac{\pi D_{closed}}{\pi D_{thin} - d_{gap}}$$

$$R = R_{thin} \left(\frac{\pi D_{closed}}{\pi D_{thin} - d_{gap}} \right)^2 \cdot \frac{m_{thin}}{m_{closed}} = 0.153\text{m}\Omega$$

where we used the values $m_{thin} = 4.50\text{g} \pm 0.02\text{g}$ and $m_{closed} = 47.70\text{g} \pm 0.02\text{g}$ from our sample measurement. The measurement error for the masses depends on environmental noise, and we use 0.02g as a representative value. The rings in different experimental sets have slightly different masses, with a deviation of about 1%, which can be distinguished at the scale's level of sensitivity. Therefore, different students will measure slightly different values.

The dominant error in the mass ratio is $\Delta m_{thin}/m_{thin} = 0.5\%$. The error in πD_{closed} is 0.35%, as in part (d). After taking the square, this doubles to 0.7%. Examining eq. (1), we see that the relative error in $R_{thin}/(\pi D_{thin} - d_{gap})^2$ is the same as in R_{thin} , since the $(\pi D_{thin} - d_{gap})$ merely moves from the numerator to the denominator. The error in $R_{thin}/(\pi D_{thin} - d_{gap})^2$ is therefore 1%. Combining these three error sources, we have:

$$\frac{\Delta R}{R} = \sqrt{0.0085^2 + 0.007^2 + 0.005^2} = 1.2\%; \quad \Delta R = 0.002\text{m}\Omega$$

The distribution of sample measurement results on several different rings is consistent with this error estimate; the value $R = 0.153\text{m}\Omega$ cited above is near the bottom of the distribution.

A student who neglects d_{gap} will get a systematic error of 6%. A student who neglects the difference between D_{closed} and D_{thin} will get a systematic error of 10%. A student who neglects both, i.e. just uses $R = R_{thin}(m_{thin}/m_{closed})$, will get an error of 4%. These errors are halved if the student makes them for just one of the two factors of $\pi D_{closed}/(\pi D_{thin} - d_{gap})$.

In the above, we effectively treated the closed ring as a rectangle with length D_{closed} . It's easy to see that this introduces no errors in the mass calculation. Indeed, the precise formula for the ring's volume reads:

$$V = \pi a \left(\left(\frac{D_{closed} + w}{2} \right)^2 - \left(\frac{D_{closed} - w}{2} \right)^2 \right) = \pi D_{closed} w a$$

where a is the ring's height, and w is its width. This is the same formula as in the rectangular approximation. Some students may use this derivation in their solution.

The exact resistance calculation for a broad circular ring is more difficult, and reveals that the relative error from the rectangular approximation is $w^2/(3D_{closed}^2) = 0.6\%$. This analysis is not expected from the students, and the resulting error can be neglected with respect to the overall error of 1.5%. For completeness, we include the derivation of the exact formula:

$$\frac{1}{R} = \frac{1}{\rho} \int_{(D_{closed}-w)/2}^{(D_{closed}+w)/2} \frac{adr}{2\pi r} = \frac{a}{2\pi\rho} \ln \frac{D_{closed} + w}{D_{closed} - w} \approx \frac{wa}{\pi D_{closed}\rho} \left(1 + \frac{1}{3} \left(\frac{w}{D_{closed}}\right)^2\right)$$

where ρ is the material resistivity.

f. The student can vary z by using the screw to raise and lower the solenoid. The most precise way to measure z is simply to count the number of screw steps. The error is then $\Delta z = h/16 = 0.09\text{mm}$, where $h = 1.41\text{mm}$ is the screw step. If z is measured with the ruler, the error becomes $\Delta z = 0.5\text{mm}$, due to the ruler's resolution. A convenient point to define as $z = 0$ is when the solenoid touches the ring from above, and the screw's handle is in some fixed orientation. This point should be reproducible, either visually or by counting screw steps, in order to keep a consistent record of distances with the force measurements in the next part.

In anticipation of the force measurements, the student should place the scale under the solenoid, place the polystyrene block on the scale, and place the ring on the polystyrene block. The polystyrene block is important because it is an insulator, while the scale's platform is metallic and may alter the EMF-measuring circuit. It is also important for the quality of the force measurements, as explained below.

It is always best to start measuring from a small distance, because then the ring can be aligned with the solenoid's axis more easily. A reasonable range of z would be from $z = 0$ (near-contact with the solenoid) to $z = 5\text{cm}$. A reasonable resolution is one screw step. It can be made coarser towards large z , when the EMF variations become smaller.

The EMF can be measured by connecting the ends of the broad open ring to the voltmeter. We wish to measure the magnetic flux through the ring, and not through the rest of the circuit. To make sure that this is the case, the wires from the ring should be twisted into a braid. The EMF decreases with distance from about $\epsilon_{rms} = 22\text{mV}$ to zero, reaching about $\epsilon_{rms} = 5\text{mV}$ at $z = 5\text{cm}$. The measurement error is $\Delta\epsilon_{rms} = 0.1\text{mV}$. Sample measurement results are presented in Table 2. A plot of the measurements (with a trend line for part (h)) is presented in Graph 1.

g. The student can vary z as in part (f), this time measuring the force on the closed ring using the digital scale. Care should be taken to use the same zero point for z as in part (f). It is convenient, though not necessary, to measure the force at the exact same points where the EMF was measured previously.

Again, the ring should be placed on the polystyrene block, rather than directly on the scale. There are two reasons for this in the context of force measurements. First, the metallic parts of the scale also react to the solenoid's magnetic field. Therefore, the solenoid must be kept at a distance above the scale, to eliminate a direct effect on the scale's reading. To observe this effect and its successful elimination, the student may turn on the current in the solenoid without a ring resting on the scale, and check whether the scale's reading changes. The second reason to use the polystyrene block is the small area of the scale; if the ring rests on the scale directly, it's difficult to ensure that some of its weight doesn't fall on the scale's lid or other supporting surfaces. Finally, one must make sure that the solenoid isn't in direct contact with the ring or the solenoid block, so that its weight doesn't fall on the scale.

It is convenient to turn on the scale, or to press the Tare button, with the ring and the block resting on the scale with no current in the solenoid. We can then measure the magnetic force directly. Otherwise, the student must manually subtract the scale's reading at zero current from all of his force values.

For values of z from $z = 0$ to $z = 5\text{cm}$, the force decreases from about $\langle F \rangle = 8\text{gf}$ (grams-force) to about $\langle F \rangle = 0.25\text{gf}$. The measurement error depends on environmental noise. A representative value is $\Delta\langle F \rangle = 0.02\text{gf}$. Sample measurement results are presented in Table 2.

h. The derivative should be found from discrete differences between pairs of points, situated symmetrically around the point we are interested in. It is better to find the derivative $d\epsilon_{rms}/dz$, and then multiply it by $2\epsilon_{rms}$, than to find the derivative $d\epsilon_{rms}^2/dz$ directly. This is for two reasons. First, taking the square amplifies the errors in the discrete point differences. Second, if the student chooses a graphical method (see below), it is more convenient to use the graph of $\epsilon_{rms}(z)$: it was already drawn in part (f), and its points are more evenly distributed than the points on a graph of $\epsilon_{rms}^2(z)$.

We will now discuss two distinct methods for choosing the discrete differences for $d\epsilon_{rms}/dz$. When used properly, the two give equally good results.

Numerical method:

One method to find the derivative $d\epsilon_{rms}/dz$ is simply to take differences between measured values of ϵ_{rms} . The intervals at which the differences are taken must be carefully chosen. A small interval will give a large error in the slope, due to the statistical scatter of the measured points. On the other hand, a large interval may result in too much smearing, so that we're not capturing the local slope. An optimal interval is about 6 screw steps, i.e. about $3h = 4\text{mm}$ to each side from the point of interest.

Graphical method:

Another method is to draw a smooth trend line through the measured points, and then to use differences between points on this trend line rather than the measured values. If the measurements were carried out properly, the trend line will deviate by only $0.5\text{mm} - 1\text{mm}$ from the measured points on the graph paper. An example worked out by a hapless theoretician is shown on Graph 1. The thick line is a consequence of the trial-and-error process of sketching the best line. The discrete intervals for the derivative can now be chosen smaller than with the numerical method, since the statistical scatter is smoothed out. In particular, taking one screw step to each side from the point of interest is now good enough. One should not choose much smaller intervals, due to the limited resolution of the graph paper. A potential advantage of the graphical method is that it's not tied to the exact z values where the EMF was measured. This is helpful if the EMF and the force were not measured at the exact same heights.

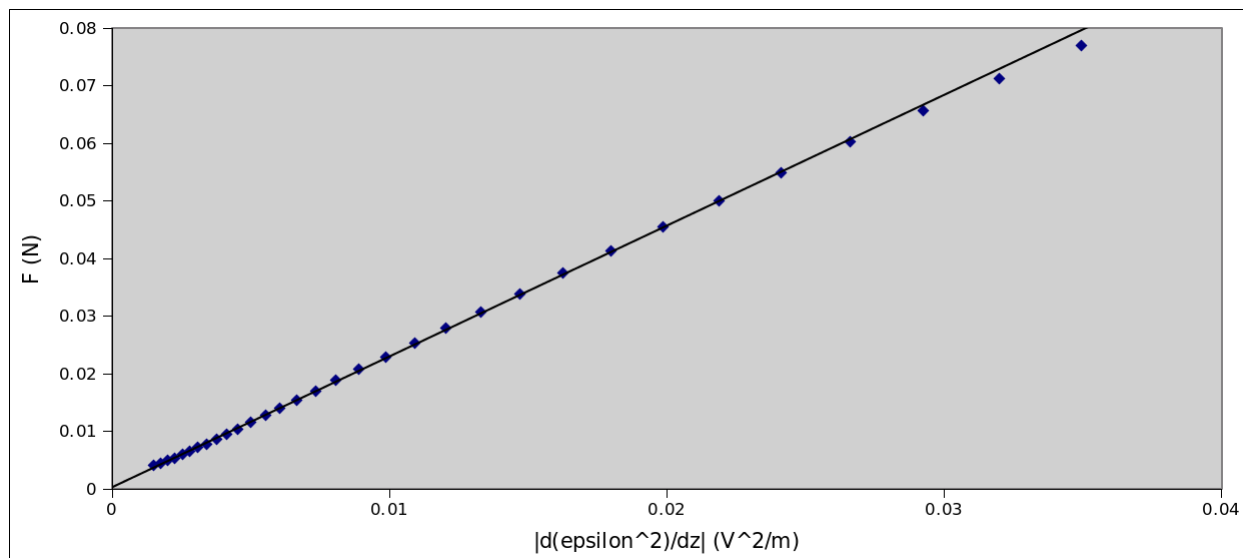
An inferior graphical method is to draw tangents to the curve $\epsilon_{rms}(z)$, and calculate the slopes of these tangents. In practice, it is very difficult to identify a tangent line visually, and using this method produces large deviations, on the order of 20%, in the analysis of part (i).

Sample values of $d\epsilon_{rms}/dz$ and $d\epsilon_{rms}^2/dz$ are provided in Table 2.

i. Following parts (c) and (h), the student should draw a linear graph of $\langle F \rangle$ as a function of $d\epsilon_{rms}^2/dz$. The graph should pass through the origin, and its slope equals:

$$k = \frac{L}{2(R^2 + \omega^2 L^2)} \quad (2)$$

If very small distances are included in the graph (under about 6mm between the ring and the solenoid's edge), the corresponding points will deviate from linearity, with a visible decrease in the slope. This can be seen in the computerized plot below:



This effect results from attraction between the ring and the solenoid's iron core.

In the manual graph from our sample experiment (Graph 2), the non-linearity isn't observed, and the slope comes out $k = 2.29 \text{ Nm/V}^2 \pm 0.04 \text{ Nm/V}^2$ (2%). To extract L , we must solve the quadratic equation:

$$2\omega^2 k L^2 - L + 2kR^2 = 0$$

The two roots are:

$$L = \frac{1 \pm \sqrt{1 - 16\omega^2 k^2 R^2}}{4\omega^2 k}$$

where we must use $\omega = 2\pi \cdot 50 \text{ Hz} \approx 314 \text{ Hz}$.

The student can find, either analytically or by substitution, that only the smaller root satisfies $\omega L < R$. In our sample experiment, the result reads:

$$L = \frac{1 - \sqrt{1 - 16\omega^2 k^2 R^2}}{4\omega^2 k} = 1.13 \cdot 10^{-7} \text{ H} = 0.113 \mu\text{H}$$

If the student hasn't done so before this point, he will need convert the force units from grams-force to Newtons using the provided value of g .

We find that the ratio $\omega L/R$ is 0.23. Therefore, for the error estimation, we can write eq. (2) as:

$$k = \frac{L}{2R^2} \Rightarrow L = 2kR^2 \quad (3)$$

As we can see, the delicate error considerations in R are now even more important, since it appears squared. Collecting the relative errors in k and R , we have:

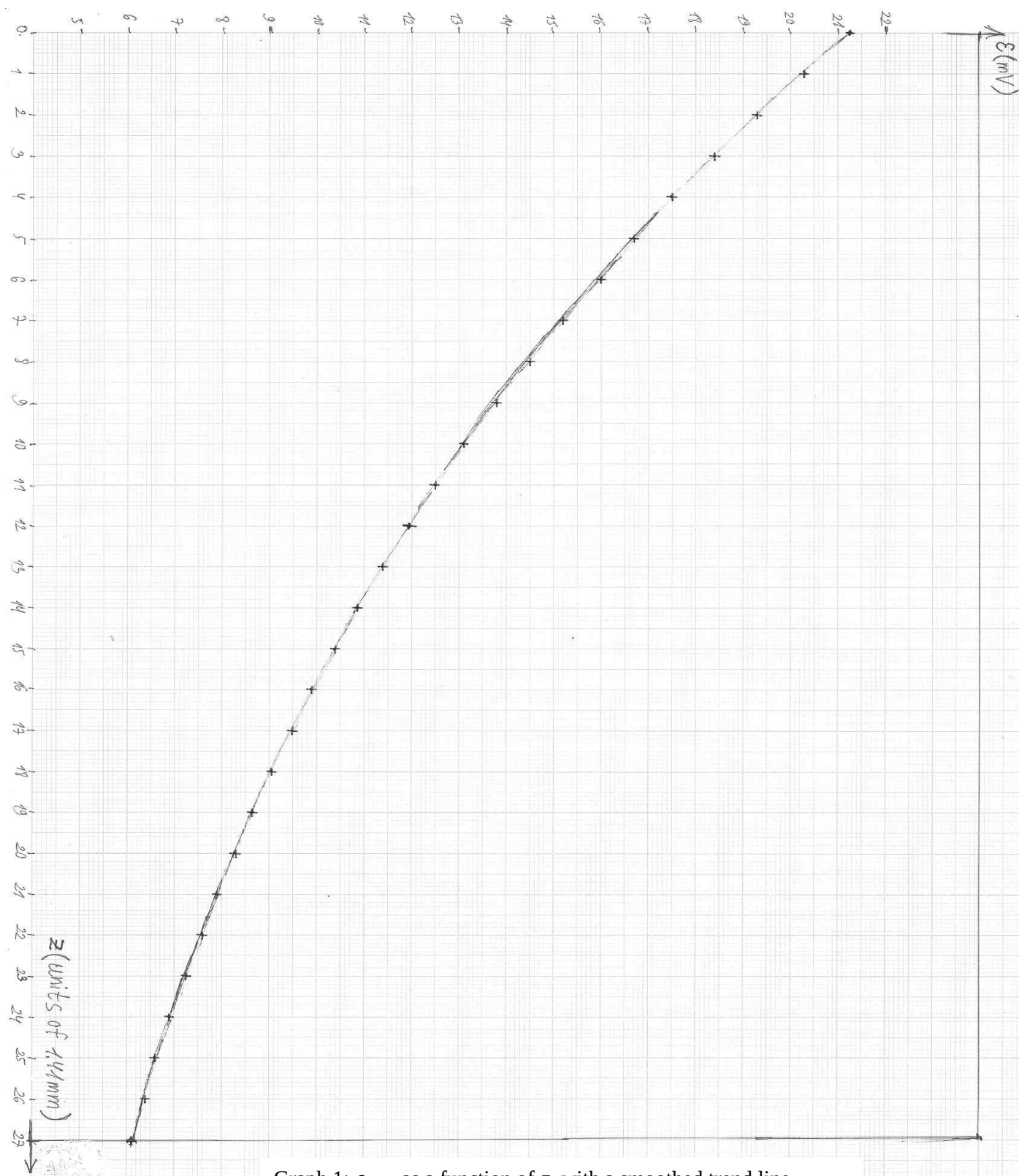
$$\frac{\Delta L}{L} = \sqrt{0.02^2 + (2 \cdot 0.012)^2} = 0.03 = 3\%; \quad \Delta L = 0.003\mu\text{H}$$

The scatter of results from sample experiments is consistent with this error estimation.

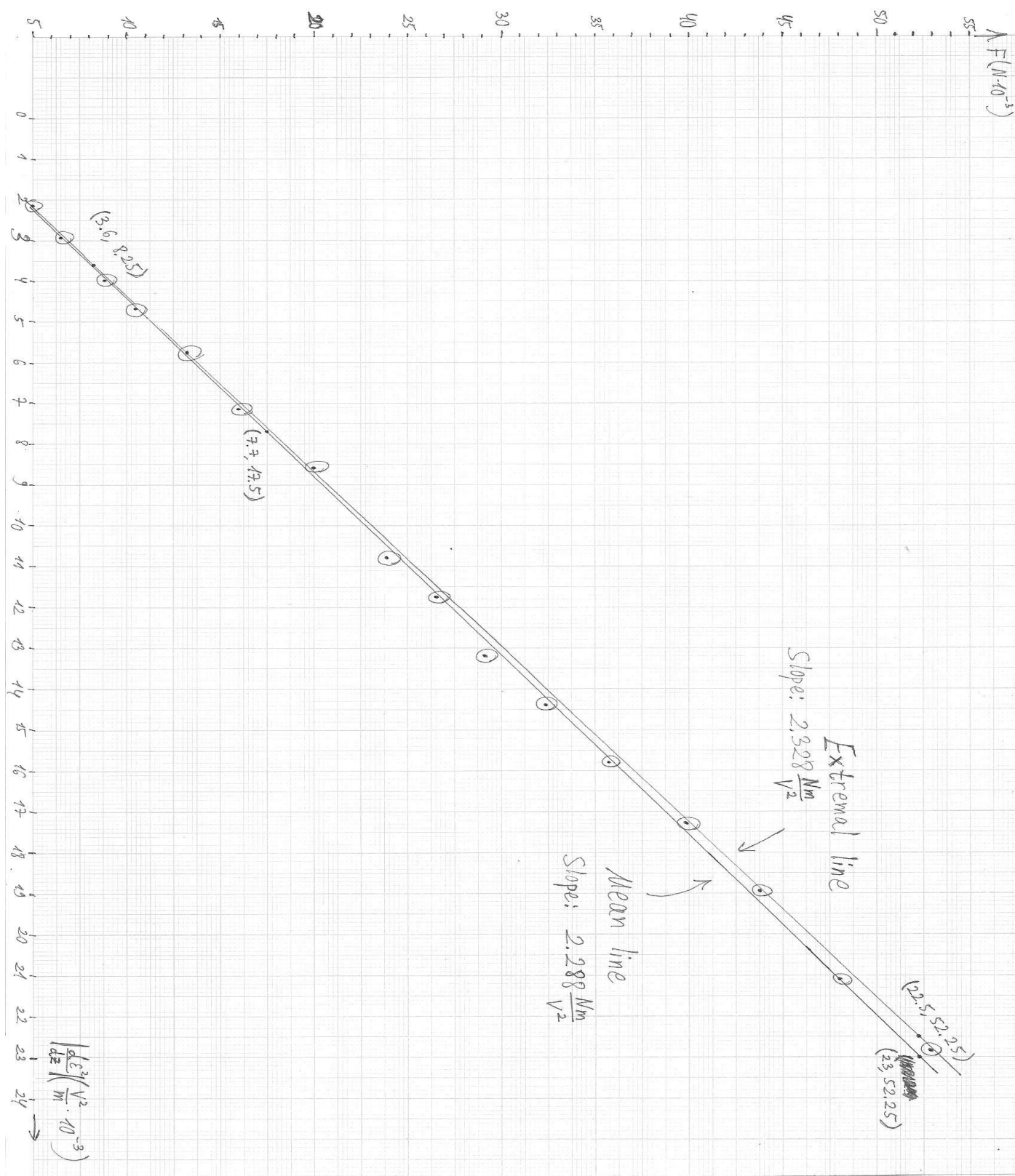
A student who neglects $(\omega L/R)^2$ all along and uses eq. (3) to find the *value* of L will introduce a systematic error of 5%.

n (screw steps) ± 0.1	z (mm) $\pm$ 0.15mm	ϵ_{rms} (mV) $\pm$ 0.1mV	$\langle F \rangle$ (gf) $\pm$ 0.02gf	$\langle F \rangle$ (N) $\pm$ 0.0002N	$ d\epsilon_{rms}/dz $ (V/m)	$ d\epsilon_{rms}^2/dz $ (V ² /m)
0	0	21.25	7.08	0.06938		
1	1.41	20.3	6.5	0.06370		
2	2.82	19.3	5.93	0.05811		
3	4.23	18.4	5.4	0.05292	0.6206	0.02284
4	5.64	17.5	4.9	0.04802	0.6028	0.02110
5	7.05	16.7	4.47	0.04381	0.5674	0.01895
6	8.46	16	4.07	0.03989	0.5437	0.01731
7	9.87	15.2	3.65	0.03577	0.5201	0.01581
8	11.28	14.5	3.3	0.03234	0.4965	0.01440
9	12.69	13.8	2.97	0.02911	0.4787	0.01321
10	14.1	13.1	2.71	0.02656	0.4492	0.01177
11	15.51	12.5	2.44	0.02391	0.4314	0.01079
12	16.92	11.95	2.2	0.02156	0.4019	0.00961
13	18.33	11.4	2.045	0.02004	0.3783	0.00862
14	19.74	10.85	1.83	0.01793	0.3546	0.00770
15	21.15	10.4	1.64	0.01607	0.3428	0.00713
16	22.56	9.9	1.45	0.01421	0.3251	0.00644
17	23.97	9.5	1.35	0.01323	0.3014	0.00573
18	25.38	9.05	1.2	0.01176	0.2955	0.00535
19	26.79	8.65	1.07	0.01049	0.2719	0.00470
20	28.2	8.3	1	0.00980	0.2660	0.00442
21	29.61	7.9	0.905	0.00887	0.2541	0.00402
22	31.02	7.6	0.815	0.00799	0.2423	0.00368
23	32.43	7.25	0.74	0.00725	0.2246	0.00326
24	33.84	6.9	0.67	0.00657	0.2128	0.00294
25	35.25	6.6	0.61	0.00598	0.2009	0.00265
26	36.66	6.4	0.56	0.00549	0.1950	0.00250
27	38.07	6.1	0.52	0.00510	0.1773	0.00216
28	39.48	5.9	0.47	0.00461		
29	40.89	5.6	0.42	0.00412		
30	42.3	5.4	0.36	0.00353		

Table 2: Sample EMF and force measurements and derivative values



Graph 1: ϵ_{rms} as a function of z , with a smoothed trend line.



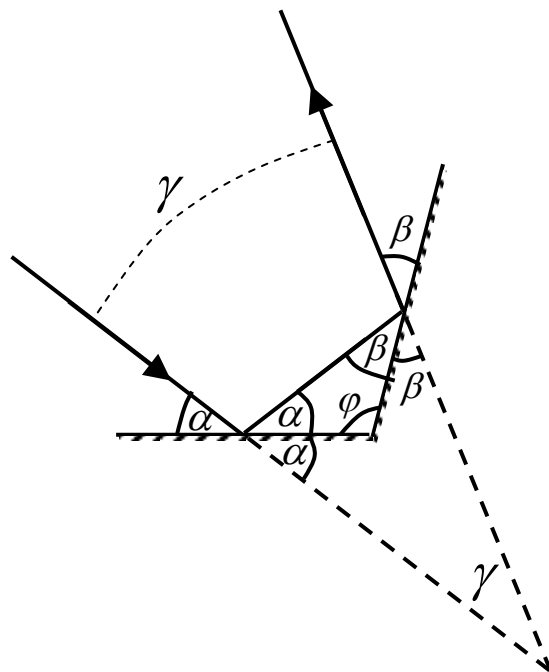
Graph 2: $\langle F \rangle$ as a function of $|d\epsilon_{rms}^2/dz|$, with linear trend lines.

Experimental Question 2: An Optical "Black Box"

SOLUTION

a. Consider an incoming light beam at angle α to the first mirror (see figure). The angle to the second mirror is given by $\beta = \pi - \varphi - \alpha$. Thus, the total deflection angle is $2\alpha + 2\beta = 2(\pi - \varphi)$. In particular, it is independent of α . The angle γ is then given by:

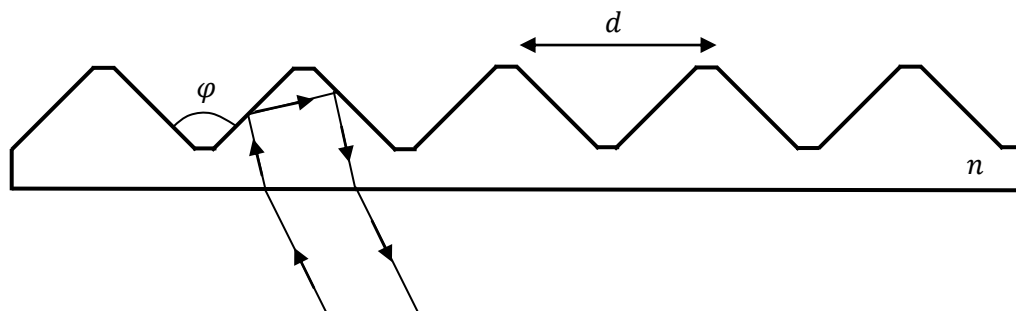
$$\gamma = \pi - 2(\pi - \varphi) = 2\varphi - \pi$$



b. The correct option is A. Options C and D are ruled out, because both the transmission and reflection patterns possess left-right symmetry. Option B is ruled out, because it implies strong forward transmission at perpendicular illumination; instead, strong transmission is observed only to the sides, as described in question (d).

c. To determine φ , we must observe the reflection pattern, using the setup suggested in the question. It is important to place the flashlight into the hole in the screen; otherwise, the observed pattern will be smeared due to the sample's non-zero size.

A strong reflection pattern is observed when the light enters the sample from its flat side, is reflected twice from the two slanted faces, and exits again from the flat side. This optical path results in total internal reflection, producing a much brighter reflection pattern than the pattern for light coming from the "toothed" side.



The observed pattern is a colorful band of interference fringes, gradually fading to the sides, without well-defined edges. As we rotate the sample, it can be seen that the pattern widens and narrows, while its center remains stationary, directly on top of the flashlight's center. While the color sequence is complicated, it is visibly symmetric around the center, which appears white; these observations are the key to identifying the center precisely.



The center of the pattern corresponds to the optical path due to geometric optics. From part (a), we see that such a zero reflection angle corresponds to an angle $\varphi = \pi/2$ between the two reflecting surfaces. This conclusion is supported by two other qualitative observations. First, if the reflection angle were nonzero, light from different areas of the sample would reach different points on the screen, due to the sample's non-zero size. This would have led to a smeared reflection pattern, like the one obtained when the flashlight does not coincide with the screen. Second, we see that the location of the pattern's center doesn't depend on the sample's orientation. Without refraction at the flat boundary, this would have been the case for any reflection angle. However, taking the refraction into account, the angle between the incoming and outgoing rays would have to be a function of the incidence angle, *except* for the case $\varphi = \pi/2$, for which the angle remains zero after refraction (see figure above).

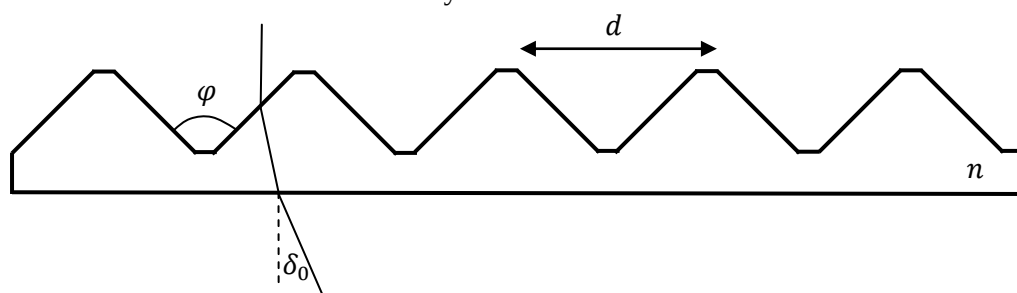
The center of the interference pattern can be identified with an error of $\Delta x = 0.5\text{mm} - 1\text{mm}$ for the optimal observation distance $y \approx 50\text{cm}$ between the flashlight and the sample. This results in an error of at most 0.002 radians in the angle between the incoming and outgoing rays. Examining the result of part (a), we see that this error should be halved to give the error in φ . Though it is not required, a bright student will also divide the error by the refraction index n , which can be estimated as ~ 1.5 . This is because the refraction amplifies angular deviations as the light leaves the sample. We conclude that the error in φ is $\Delta\varphi = 7 \cdot 10^{-4}\text{rad} = 0.04^\circ$. This will be negligible with respect to all other errors in the experiment.

A common mistake will be to take the edge of the central white stripe, or the edge of the entire pattern, as representing φ instead of the pattern's center. For the edge of the central white stripe (about $x = 5\text{mm}$ at $y = 50\text{cm}$), this would result in a $0.15^\circ - 0.3^\circ$ deviation (depending on whether n is taken into account). For the edge of the pattern (about $x = 3\text{cm} - 5\text{cm}$ at $y = 50\text{cm}$), this would result in a $1^\circ - 3^\circ$ deviation.

Similarly, a student may decide that the edge of the central white stripe or the edge of the pattern correspond to the error $\Delta\varphi$. This will lead to an over-estimation of the error as $0.15^\circ - 0.3^\circ$ or $1^\circ - 3^\circ$, respectively.

d. The correct option is C. An interference pattern will always have larger deflection angles for larger wavelengths. Therefore, the pattern in Figure 7 cannot result from interference, and must result from the dependence of n on the wavelength. On the other hand, the pattern in Figure 6 has the correct order of colors for an interference pattern. It must in fact be an interference pattern, because the sample's structure as depicted in part (b) is rather simple and periodic, and cannot produce a large number of nearby refraction angles.

e. The phenomenon described in this part is observed when the sample is illuminated from its "toothed" side. Transmission from the other side is very weak due to total internal reflection.



The deflection angle δ_0 can be seen qualitatively by projecting the transmission pattern onto the screen. However, due to the sample's nonzero size, this pattern is too smeared to enable precise measurement. A much sharper pattern is observed when looking directly through the sample, as suggested in the question. The challenge is then to measure angles that are seen directly with the eye, rather than on a screen.

For this purpose, the student is provided with the mobile wooden stake on a bench. The bench should be placed perpendicularly to the line of sight between the flashlight and sample. The stake should be positioned at two points: point A directly along the line of sight between the flashlight and sample, and point B where the stake coincides with the desired band of deflected light along the student's line of sight. The two measurements are the distance $x = AB$ between these two points and the distance y between the sample and the center of the bench. The desired deflection angle is then given by:

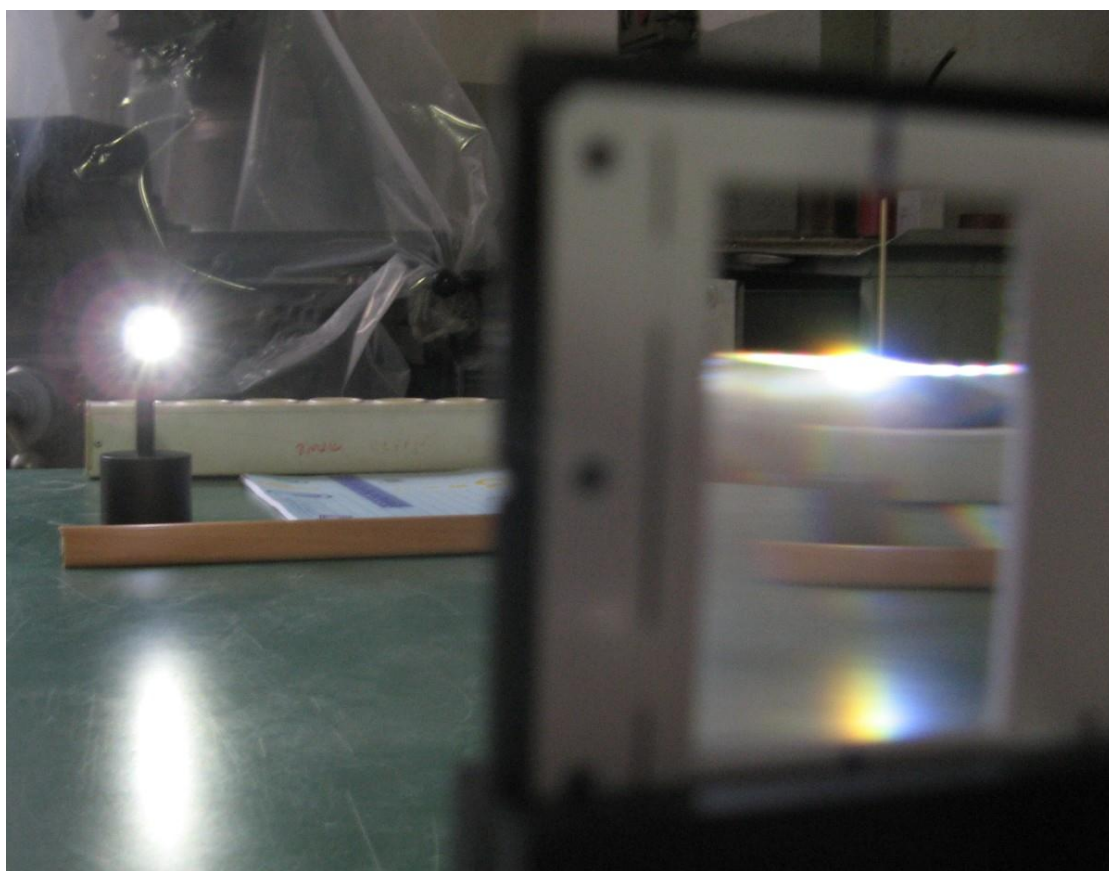
$$\delta_0 = \tan^{-1} \frac{x}{y}$$

To properly establish point A, some alignment issues must be solved. The long bench can be quite accurately placed in parallel to one of the desk's sides. The flashlight can then be placed near the bench, in perpendicular to it. The sample can also be aligned in parallel to the desk's side, by visual reference to the orientation of its long handle. The bench should lie in front of the flashlight, rather than behind it. This has two advantages. First, it allows for a greater distance between the flashlight and sample, minimizing the harm from the sample's nonzero size. Second, it allows us to align the stake directly in front of the flashlight quite accurately. The alignment of the stake and flashlight can be checked and corrected by observing the stake's shadow on the desk. Finally, we must place the sample precisely in front of the flashlight, i.e. without a sideways offset. One way to do this is to place both the flashlight and sample at the edge of the desk. A less accurate way is to measure their distances from the edge using the tape measure. In any case, there are several ways to check the alignment without reference to the desk:

1. Ensuring that the sample lies at the center of the stake's shadow from the flashlight.
2. Looking at the flashlight through the sample's frame, and making sure that the flashlight's center and the stake are aligned together with the frame's center.

3. Placing a sheet of paper behind the sample, and making sure that the stake's shadow on the paper is at the center of the shadow from the sample's frame.

To establish point B, it is best to look through the sample not from a zero distance, but from a distance large enough to focus the eye on the sample's frame. One should then try to align three points along the line of sight: the center of the sample's frame (marked with a blue line), the desired band of deflected light, and the stake. It is convenient first to align the frame's center with the deflected light, and then check in which direction the stake should be moved to align with them. Within 3-5 such iterations, point B can be established with an accuracy of about $\pm 2\text{mm}$.



Once the points A and B are physically established, the distance between them has to be measured. One way is to record each position of the stake with a pen mark on the bench, and then measure the distance between the two markings. It's difficult and unnecessary to mark the position of the stake itself – one can mark the position of the edge of the stake's holder. Another way, which doesn't require any markings, is to align the edge of the stake's holder at point A with the edge of the bench. Then when the stake is at point B, the distance x can be measured directly. In any case, while moving the stake between points A and B, the student must take care not to move the bench.

The relative accuracy of the measurement increases with the distance between the flashlight and bench and the sample. Optimally, most of the desk's 1-meter length should be used for the distance y . Then x will take up most of the desk's width. The price for these large distances is that the bench is not within arm's reach from the sample, and the student will have to get up for each adjustment of the stake's position.

The cylindrical base of the sample holder makes it difficult to measure the distance y accurately. The solution is to place the measuring tape perpendicularly to the bench or screen, and in tangent to the cylindrical base. Repeated measurements reveal that this reduces the error to 1mm.

We now present the results of a sample measurement:

$$x = 471\text{mm} \pm 2\text{mm}, \quad y = 775\text{mm} \pm 1\text{mm}$$

From these we derive the deflection angle $\delta_0 = \tan^{-1}(x/y) = 31.3^\circ$. The error in δ_0 as derived from the measurement errors reads:

$$\Delta\delta_0 = \sqrt{\left(\frac{\Delta x/y}{1 + x^2/y^2}\right)^2 + \left(\frac{x\Delta y/y^2}{1 + x^2/y^2}\right)^2} = 0.002\text{rad} = 0.1^\circ \quad (1)$$

One can also use the simpler formula $\Delta\delta_0 \approx \Delta(x/y)$ to arrive at a similar result. The value $\Delta x = 2\text{mm}$ leading to this estimate was obtained from repeated attempts to locate point B without changing the overall alignment. A student who repeats the alignment process from the beginning and compares his results will likely get a larger error estimate, up to $\Delta\delta_0 = 0.3^\circ$. The source of these deviations is the difficulty in arranging perpendicular incidence of the light onto the sample. This error is difficult to quantify in any way other than repeating the alignment process. An additional source of error is the subjective definition of the “blue end of the visible spectrum”. Sample measurements were performed by different individuals, and the scatter of their results is taken into account in the Marking Scheme.

Instead of taking the distance between the source and the deflected light band, a student may take the distance between the two symmetric deflected bands on either side of the source. This carries two disadvantages: the size of the desk constrains the perpendicular distance y in this case to about 40cm, and the difficulties with locating point B are now encountered twice.

Another inferior alternative is to use the screen’s edges or center instead of the wooden stake. Then the distance x is fixed by the screen’s width, and y becomes the controlled variable. This method leads to higher errors (about 0.5°), for two reasons. First, the stake is helpful in the alignment of the different elements, as described above. Second, it is difficult to fine-tune the distance y without spoiling the alignment.

f. In this part, after the initial alignment and before measuring the deflection angle, the student should gently rotate the sample while looking through it, until he identifies the orientation at which the deflection angle is minimal. Unlike the deflection of perpendicular light in part (e), this minimal deflection angle can be observed from both sides of the sample. When the light is incident on the “toothed” side, as in part (e), the minimum is obtained not far from perpendicular incidence. As a result, $\delta_{\min}$ is only slightly smaller than δ_0 .

The measurement process is very similar to that in part (e), except that there’s no need to align the sample for perpendicular incidence. The student may keep the rest of the alignment from part (e), and use the same distance y between the sample and the wooden bench. The results of a sample measurement read:

$$x = 461\text{mm} \pm 2\text{mm}, \quad y = 775\text{mm} \pm 1\text{mm}$$

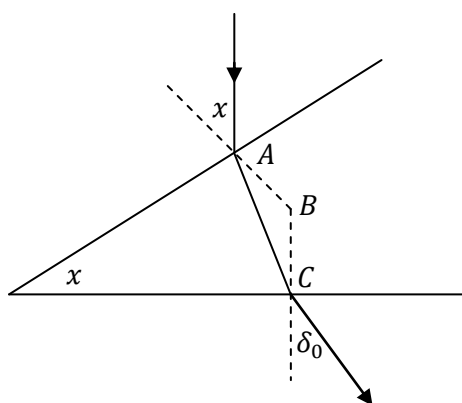
From which we derive:

$$\delta_{min} = \tan^{-1}(x/y) = 30.75^\circ \pm 0.1^\circ \quad (2)$$

In this case, the naïve error estimation of eq. (1) corresponds to the actual scatter of results from repeated alignments and measurements. This is because the error from the perpendicular alignment is gone. We still have an error in finding the sample orientation which leads to the minimal deflection. However, this error has very little impact on the measurement of δ_{min} itself, since the deflection angle is stationary around the minimum.

g. Solution in terms of δ_0 :

Consider a vertical light ray entering a horizontal prism with opening angle x (see figure). We have:



$$\begin{aligned} \sin \angle BAC &= \frac{\sin x}{n} \\ \angle ABC &= \pi - x \\ \angle BCA &= \pi - \angle BAC - \angle ABC = x - \angle BAC \\ \sin \delta_0 &= n \sin \angle BCA = n \sin(x - \angle BAC) = \\ &= n(\sin x \cos \angle BAC - \cos x \sin \angle BAC) = \\ &= n \left(\sin x \sqrt{1 - \left(\frac{\sin x}{n}\right)^2} - \cos x \frac{\sin x}{n} \right) = \\ &= \sin x \left(\sqrt{n^2 - (\sin x)^2} - \cos x \right) \end{aligned}$$

Solving for n , we get:

$$n = \sqrt{(\sin x)^2 + \left(\cos x + \frac{\sin \delta_0}{\sin x} \right)^2}$$

In our case, the appropriate prism angle is $x = (\pi - \varphi)/2$, which gives:

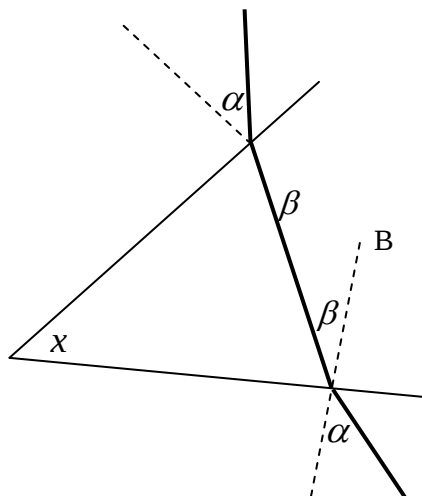
$$n = \sqrt{(\cos(\varphi/2))^2 + \left(\sin(\varphi/2) + \frac{\sin \delta_0}{\cos(\varphi/2)} \right)^2}$$

A student that plugs in the value $\varphi = \pi/2$ will get:

$$n = \sqrt{\frac{1 + (1 + 2 \sin \delta_0)^2}{2}}$$

Solution in terms of δ_{min} :

Using the uniqueness of the minimal deflection angle δ_{min} and the reversibility of light rays, we conclude that minimal deflection occurs symmetrically with respect to the prism angle:



$$\begin{aligned}\angle B &= \pi - x \\ \beta &= \frac{\pi - \angle B}{2} = \frac{x}{2} \\ \delta_{min} &= 2(\alpha - \beta) = 2\alpha - x \\ n \sin \frac{x}{2} &= n \sin \beta = \sin \alpha = \sin \frac{x + \delta_{min}}{2}\end{aligned}$$

Solving for n , we get:

$$n = \frac{\sin \frac{x + \delta_{min}}{2}}{\sin \frac{x}{2}}$$

In our case, the appropriate prism angle is $x = (\pi - \varphi)/2$, which gives:

$$n = \frac{\sin \frac{\pi - \varphi + 2\delta_{min}}{4}}{\sin \frac{\pi - \varphi}{4}}$$

A student that plugs in the value $\varphi = \pi/2$ will get:

$$n = \frac{\sin \frac{\pi + 4\delta_{min}}{8}}{\sin \frac{\pi}{8}} \quad (3)$$

h. The refraction index n_v can be found either from δ_0 or from δ_{min} , using the formulae from part (g). As discussed in parts (e) and (f), the measurement of δ_{min} is the more accurate one. Therefore, its accuracy is the basis for the full credit in the Marking Scheme for this part. Using the results (2) and (3), we find for our sample experiment:

$$n_v = 1.604$$

The error reads:

$$\Delta n_v = \frac{\cos \frac{\pi + 4\delta_{min}}{8}}{2 \sin \frac{\pi}{8}} \Delta \delta_{min} = 0.002 \quad (4)$$

i. The measurement process here is the standard one for a diffraction grating. The pattern is projected onto the screen. The measured quantities are the distance y between the sample and the screen, the distances x on the screen between the fringes and the pattern's center, and the fringe number m . The angles θ for the different fringes are then found as $\theta = \tan^{-1}(x/y)$. It is important that x is measured from the center, since the relation $d \sin \theta = m\lambda$ is linear in $\sin \theta$ and not in $x \sim \tan \theta$. An offset in m is less problematic, since it will not affect the slope of the linear graph in part (j).

The alignment process is more straightforward than with the white light. The laser beam, observed as scattered light from the screen or from the sample, can be used as a reference line instead of the stake's shadow. The sample's nonzero size no longer leads to measurement uncertainties, since the laser beam illuminates only a small portion of it. Instead, we are constrained by the width of the beam, which widens noticeably with distance. Since the laser pointer is not quite horizontal, the beam may come out at a slight vertical slope, causing it to miss the sample. To fix this, the student may rotate the laser around its axis.

A diffraction pattern can be observed in three different configurations:

1. Reflection with the light entering the sample from the flat side, as in part (c).
2. Transmission with the light entering the sample from the "toothed" side, as in parts (d)-(e).
3. Transmission with the light entering the sample from the flat side.

All three configurations produce fringes at the same angles $\theta(m)$, but otherwise they look quite different. Each configuration has its advantages and disadvantages for our measurement purposes. The best choice among the three may depend on the amount of ambient light in the room.

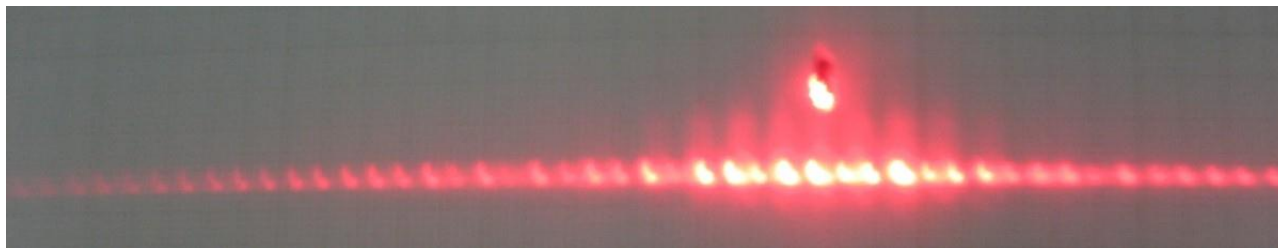
Configuration 3 gives a wide pattern of dim but well-defined fringes. They result from the interference of light passing through the narrow flattened "peaks" and "valleys" of the sample's toothed side (the light incident on the slanted faces of the "teeth" undergoes total internal reflection). Note: the observed patterns are sharper than in the photos below, which were taken by a theoretician.



Configuration 2 gives a similar pattern from the "peaks" and "valleys", superimposed on two much brighter narrow interference patterns from the slanted faces. These bright patterns are centered around the geometric deflection angle δ_0 , one to the right and one to the left. They are bright because more light passes through the slanted faces, and they are narrow because they are multiplied by the narrow diffraction pattern off of each slanted face, whose width is the same order of magnitude as the spacing d . The condition $d \sin \theta = m\lambda$ for a fringe at angle θ is the same for the "peaks" and "valleys" and for the slanted faces.



Configuration 1 (reflection) gives a noisier interference pattern, with broader fringes. This is because the laser beam travels the distance to the screen twice (back and forth), and broadens in the process. The central fringes are very bright – most of the light is reflected off of both slanted faces directly backwards, as discussed in the solution to part (c). These fringes are even brighter than the bright patterns in Configuration 2, because in this case the beam doesn't split in two. This bright region of the reflected pattern is narrow, again due to the narrow diffraction pattern off of the broad slanted faces. Further to the sides, the dim interference pattern from the "peaks" and "valleys" becomes dominant. It is dimmer than in the transmission patterns, and fades away quickly, because it arises from weak reflection, as opposed to the total internal reflection from the slanted faces.



The dim interference pattern from the narrow “peaks” and “valleys” on the sample displays an alternating sequence of brighter and dimmer fringes, as described in the question text. If all the fringes are taken into account, their spacing corresponds to the sample’s spacing d . The twice larger spacing between the *brighter* fringes corresponds to the spacing $d/2$ between each “peaks” and the adjacent “valley”. In other words, the dimmer fringes are due to destructive interference between a peak and its adjacent valley. Towards larger θ , the effect becomes less simple than just a factor-of-2, because then the vertical distance between the “peaks” and “valleys” must also be taken into account. As the sample is rotated, the brighter and dimmer fringes interchange, and at some orientations become equal in magnitude. This phenomenon is more pronounced in the transmission pattern, i.e. Configurations 2 and 3. In the reflection pattern, i.e. Configuration 1, it is pronounced towards the edges, where the light from the “peaks” and “valleys” becomes dominant. The central fringes in Configuration 1 remain fixed up to small changes in brightness, and their spacing corresponds to d . The bright side fringes in Configuration 2 vary noticeably in brightness, due to the diffraction pattern from each slanted face.

It is likely that some of the dimmer fringes in the alternating sequence will not be visible. This results in a doubled, and sometimes tripled, spacing between the visible fringes. By rotating the sample and observing the alternating pattern, the student should learn to identify such doubled spacing, and take it into account when recording the fringe numbers m .

In the reflection setup, rotation of the sample reveals a bright fringe moving across the screen. This fringe is of similar brightness to the central fringes of the interference pattern. It arises from light directly reflected from the sample’s flat side, without entering into the sample. The other bright fringes remain stationary as this fringe moves, because they arise from a double reflection, as discussed in parts (a) and (c). The single-reflection fringe can be used as an alignment indicator: at perpendicular illumination, it should fall back on the laser. This situation can only be extrapolated and not directly observed, because near the center the single-reflection fringe blends with the other bright fringes.

The reflection pattern has an advantage for measurements – the central fringes around $m = 0$ are very bright and always visible, so there is no problem with counting the fringes consecutively, even in bright ambient illumination. On the other hand, it has some disadvantages – the overall pattern is narrower, while the individual fringes are broader. The reflection setup is also more difficult to align: to observe a high-quality pattern, the laser beam must pass through the hole in the screen without scattering off the sides, and then it must reach the sample. On the other hand, this provides better feedback: if a clean pattern is obtained, it means that the alignment is good. This reduces the eventual error.

In all three configurations, we need an indicator for the point $x = 0$, from which both x and m should be counted. In Configuration 1 (reflection), given proper alignment, this point is given by the source of the laser beam, at the center of the hole in the screen. In Configurations 2 and 3 (transmission), it can be found by illuminating the screen directly without the sample, and recording the beam’s location. Configuration 2 allows another method – to define $x = 0$ as

the midpoint between the two bright side patterns. See the solution to (k) for remarks on identifying the centers of these patterns.

When choosing the measurement points, one should choose the largest possible range for which the fringes can still be reliably counted. This will reduce the relative error in distance measurements. There is no need to record all the dozens of fringes in this range – a sample of 8 more-or-less evenly spaced points is enough.

In the white light measurements, it was clear that one must choose the largest possible distance y to minimize the relative errors. With the laser light, a compromise must be struck between this consideration and the widening of the laser beam with distance.

In Table 1, we present sample measurements from a reflection pattern at a distance $y = 652\text{mm} \pm 1\text{mm}$. The error in x can be understood in two different ways. First, one may consider the error due to the width of each separate fringe. Since we are looking for the center of the fringes, this error is smaller than the fringe width itself. It may be estimated for transmitted light as $\Delta x = 0.5\text{mm}$ (same as the measurement resolution), and for reflected light as $\Delta x = 1\text{mm}$. On the other hand, one may consider the collective offset of all the fringes relative to the true $x = 0$ point. This can be estimated as half the fringe spacing, which may be as large as $\Delta x = 1\text{cm}$. All the above error estimations should receive full credit. The corresponding errors in θ , calculated as in eq. (1), range from $\Delta\theta = 0.05^\circ$ to $\Delta\theta = 1^\circ$.

m	x (mm)	θ	$\sin \theta$
-29	-262.5	-21.93°	-0.3735
-20	-176	-15.11°	-0.2606
-10	-86.5	-7.56°	-0.1315
-1	-9	-0.79°	-0.0138
1	9	0.79°	0.0138
6	51	4.47°	0.0780
11	95.5	8.33°	0.1449
19	170	14.61°	0.2523
25	230.5	19.47°	0.3333

Table 1: Sample measurement results for part (i) from a reflection pattern at $y = 652\text{mm} \pm 1\text{mm}$

j. The student should use the relation $d \sin \theta = m\lambda$ and draw a linear graph of $\sin \theta$ as a function of m . The slope of this graph will be λ/d . The values of $\sin \theta$ for our sample experiment are shown in Table 1. Deciding on the appropriate error bars is problematic. First, we have the wide range of possible error estimations for θ from part (i). Second, it's difficult to estimate the relevance of the fringes' collective offset for the graph's slope. Therefore, error bars will not be graded.

The plot for our sample experiment is presented in Graph 1. Its slope reads:

$$k = 0.0130 \pm 0.0002 \text{ (1.5\%)}$$

This leads to the value $d = \lambda/k = 50.2\mu\text{m}$ for the spacing. The relative error in k is dominant over the relative error in λ , so we have $\Delta d = 0.8\mu\text{m}$ (1.5%).

A student who doesn't take into account the doubled spacing between some of the fringes will get a less linear graph. A student who consistently uses the double spacing will get a linear graph, but his result for d will be too small by a factor of 2.

k. The student must use the bright patterns in the transmitted light (Configuration 2 or a rotated Configuration 3) to measure the deflection angle δ_0 or δ_{min} . Instead of looking through the sample, the student must use the screen. The peculiarities of the alignment process were already discussed in part (i). If part (i) was carried out with transmitted light, one may keep the same alignment and distance y to the screen.

The precise location corresponding to the desired deflection angle can be difficult to identify on the screen. The bright patches are several centimeters wide, and their center doesn't necessarily lie on one of the interference fringes. However, there is an oval-shaped aura defining these patches, and the center can be identified as the point where this aura is broadest. Alternatively, one can try to move the interference fringes by slight rotations of the sample, and use them to probe the bright patch; for instance, its center can be estimated as the place where the brightest possible fringe is obtained.

As for white light, the more precise measurement is obtained for the minimal deflection angle δ_{min} . Unlike with the white light, a larger distance y to the screen is not necessarily better, due to the broadening of the laser beam. In our sample experiment, we chose a moderate distance $y \approx 50\text{cm}$. Then instead of using the distance x between the un-deflected beam and the beam at minimal deflection, we used the distance $2x$ between the minimal deflections to the right and to the left. These two points were obtained by rotating the sample without changing any other element of the alignment. As a check, we in fact measured the distances x_1 and x_2 of the two points from the un-deflected beam, to see that they come out similar. The difference $(x_1 - x_2)/2$ can then serve as an error estimate for x .

The measurements from the sample experiment read:

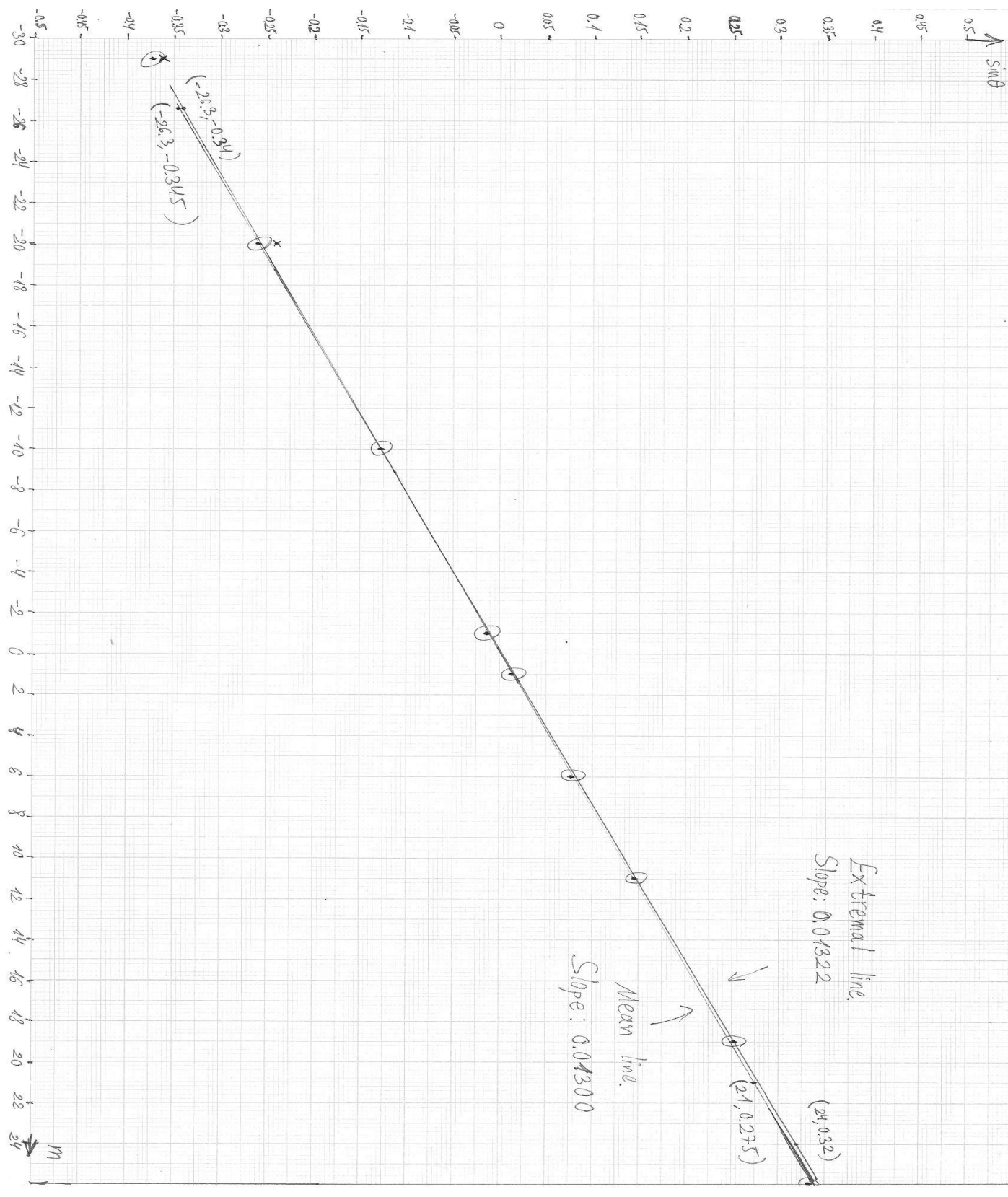
$$y = 495\text{mm} \pm 1\text{mm}; \quad x_1 = 276.5\text{mm}; \quad x_2 = 279\text{mm}$$

From x_1 and x_2 , we derive $x = (x_1 + x_2)/2 = 277.7\text{mm}$, with an error $\Delta x = (x_1 - x_2)/2 = 1.5\text{mm}$. For δ_{min} , we get:

$$\delta_{min} = \tan^{-1}(x/y) = 29.3^\circ$$

Calculating the error as in eq. (1), we get $\Delta\delta_{min} = 0.002\text{rad} = 0.1^\circ$. This error is consistent with the scatter of several sample measurements.

Using eqs. (3) and (4) for the refractive index and its error, we get $n_r = 1.578 \pm 0.002$.



Graph 1: $\sin \theta$ as a function of m , with a linear trend line.