Theoretical Question 1: The Shockley-James Paradox

In the year 1905, Albert Einstein proposed the special theory of relativity to resolve the inconsistency between Newton's mechanics and Maxwell's electromagnetism. Proper understanding of the theory led to the resolution of many apparent paradoxes. At the time, the discussion focused mostly on the propagation of electromagnetic waves.

In this question, we solve a paradox of a different type. For a fairly simple system of charges proposed by W. Shockley and R. P. James in 1967, understanding the conservation of linear momentum requires careful relativistic analysis. If a point charge is located near a magnet of changing magnetization, there's an induced electric force on the charge, but no apparent reaction on the magnet. The process may be slow enough that any electromagnetic radiation (and any momentum carried away by it) is negligible. Thus, apparently we get a cannon without recoil.

In our analysis of this system, we will demonstrate that in relativistic mechanics, a composite body may hold a non-zero mechanical momentum while remaining stationary.

Part I: Understanding the impulse on the point charge (3.3 points)

Consider a circular current loop of radius r carrying a current I_1, and a second, larger current loop of radius $R \gg r$, concentric with the first and lying in the same plane.

a. (1 pt.) A current I_2 passing through loop 2 (the larger loop) generates a magnetic flux Φ_{B1} through loop 1. Find the ratio $M_{21} = \Phi_{B1}/I_2$. It is called the mutual inductance coefficient.

b. (0.8 pts.) Given that $M_{12} = \Phi_{B2}/I_1 = M_{21}$, obtain the total induced EMF ε_2 in the larger loop as a result of a variation $\dot{I}_1 = dI_1/dt$ of the current in the smaller loop. Neglect the current in the larger loop. *Hint: the induced EMF is equal to the rate of change of the magnetic flux through the loop.*

c. (0.5 pts.) The EMF you found in part (b) is due to the tangential component of an induced electric field. Obtain an expression for the tangential electric field E at radius R as a function of the rate of change $\dot{I}_1$ of the current.

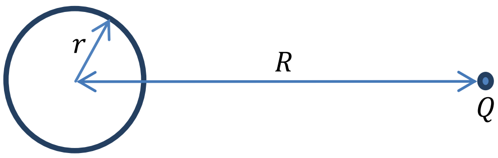

Figure 1: A circular current loop and a point charge Q.

We now remove the larger current loop, and instead put a massive point charge Q at radius R, as shown in Figure 1. It may be assumed that the charge moves very little during the relevant time periods.

d. (1 pt.) Find the total tangential impulse Δp received by the point charge as the current in the small loop changes from an initial value $I_1 = I$ to the final value $I_1 = 0$.

Part II: Understanding the recoil of the current loop (4.4 points)

We will now understand the origin of the recoil of the loop, using a loop of different geometry.

e. (1.1 pts.) Consider a hollow tube with walls made of a neutral insulating material of length l and cross section A carrying an electric current I. The current is due to charged particles of rest mass m and charge q distributed homogenously inside the tube with number density n. Assume that the charged particles are all moving along the tube with the same velocity. Find the total momentum p of the charged particles in the tube, taking Special Relativity effects into account.

f. (3.3 pts.) Consider a square current loop with side l. At a distance $R \gg l$ from the loop, there is a point charge Q; see Figure 2. The loop carries current I. We will model the current loop as a neutral tube, as in part (e). The charge carriers can move freely along the loop, colliding elastically with the walls and making elastic right turns at the corners. Neglect all interactions among the charge carriers. Assume also that all the charge carriers at a given section along the tube always move with the same velocity. Assume that the loop is heavy and that its motion can be neglected. Calculate the total linear momentum p_{hid} of the charge carriers in the loop. It is called "hidden momentum".

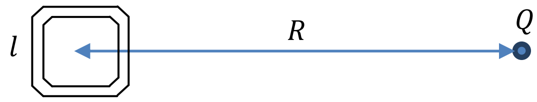

Figure 2: A square current loop and a point charge Q.

When the current stops, this linear momentum is transferred to the loop, and it gets an impulse equal to minus the impulse received by the point charge Q. This is the missing recoil that we were looking for (note that in the initial state there is also momentum in the electromagnetic field; this is important for conservation of the total momentum of the entire system).

Part III: Summarizing the results (2.3 points)

g. (0.8 pts.) Current loops are often characterized by their magnetic moment $\mu = IS$, where I is the current and S is the loop's area. Express the answer to part (d) in terms of μ, r, R and Q. Likewise, express the answer to part (f) in terms of μ, l, R and Q. Note that the electric and magnetic constants are related by:

$$\frac{4\pi k}{\mu_0} = \frac{1}{\varepsilon_0 \mu_0} = c^2$$

where c is the speed of light.

h. (1.5 pts.) In a more realistic model, the current loop is a conducting wire, and the field of the point charge Q does not penetrate into the conductor. We assume that the current is still conducted by charge carriers inside the wire. Decide whether each of the following statements is true or false, and circle the correct option in the Answer Form. **Note:** You may leave a statement undecided, but if you decide incorrectly, you will not get credit at all for part (h).

A. (0.5 pts.) The linear momentum of the current loop is zero.

B. (0.5 pts.) As the total current in the loop changes from I to zero, the charge carriers decelerate, causing induced currents in the wire's conducting material. Because of these induced currents, the point charge Q will not get a net impulse.

C. (0.5 pts.) The surface charges on the wire, induced by the presence of the external charge, will experience an electric force as the current changes from I to zero. This way, the loop will get the same impulse as found in part (f).

Theoretical Question 2: Creaking Door

The phenomenon of creaking is very common, and can be found in doors, closets, chalk squeaking on a blackboard, playing a violin, new shoes, car brakes and other systems from everyday life. Here in Israel, a similar phenomenon causes violent earthquakes with a period of several decades. These originate in the Dead Sea rift, located right above the deepest known break in the earth's crust.

The physical mechanism for creaking is based on elasticity combined with the difference between the static and the kinetic friction coefficients. In this question, we will study this mechanism and its application to the case of an opening door.

Part I: General model (7.5 points)

Consider the following system (see Figure 1):

A box with mass m is attached to a long ideal spring with spring constant k, whose other end is driven at a constant velocity u. The static and the kinetic friction coefficients between the box and the floor are given respectively by μ_s and μ_k, where $\mu_k < \mu_s$.

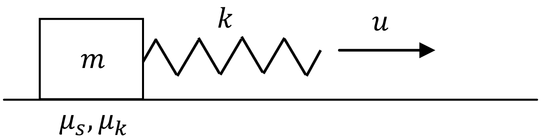

Figure 1: A general model for creaking

We would like to understand why this setup supports two different forms of motion:

1. The friction is always kinetic. This is known as a **pure slip** mode.
2. Kinetic and static friction alternate. This is known as a **stick-slip** mode. Stick-slip motion is the source of the creaking sound commonly encountered.

a. (1 pt.) Consider the case where at the initial time $t = 0$, the box slides on the floor with velocity v_0, and the spring's tension exactly balances the kinetic friction. Assume $0 < v_0 < u$. The **spring's elongation** x will oscillate as a function of t.

a1. (0.6 points) Find the period T_0 and the amplitude A of these oscillations.

a2. (0.4 points) Sketch a qualitative graph of the spring's elongation $x(t)$ for $0 < t < 3T_0$.

b. (1.2 pts.) Now, consider the case where at $t = 0$ the box is at rest, while the initial spring elongation x is the same as in part (a). Sketch a qualitative graph of the velocity $v(t)$ of the box **with respect to the floor** for $0 < t < 3T$, where T is the (new) period of the oscillations $x(t)$. Motion to the right corresponds to a positive sign of v. Indicate approximately on your graph the horizontal line $v = u$.

c. (0.5 pts.) For the initial conditions of part (b), find the time-averaged value $\bar{x}$ of the spring's elongation after a sufficiently long time has passed.

d. (2.4 pts.) For the conditions of part (b), find the period T of the oscillations $x(t)$.

Generically, stick-slip motion stops at high driving velocities u. We will now discuss one of the possible mechanisms behind this effect.

e. (2.4 pts.) Suppose that during each period T, a small amount of energy is dissipated into heat in the spring, via an additional mechanism. Let $\eta = |\Delta A/A|$ be the fractional amplitude loss per period due to dissipation in pure-slip motion. For $\eta \ll 1$, find the critical driving velocity u_c above which periodic stick-slip becomes impossible.

The results of part (e) are not required for part II.

Part II: Application to creaking door (2.5 points)

A door hinge is a hollow, open-ended metal cylinder with radius r, height h and thickness Δr. The lower end of the cylinder lies on a metal base attached to the wall (the area of contact is a ring of radius r); see Figure 2. The static and the kinetic friction coefficients between the cylinder and its base are μ_s and μ_k respectively, with $\mu_k < \mu_s$. The upper end of the cylinder is attached to the door, which can be regarded as perfectly rigid. A typical door hangs on two or three such hinges, but its weight is concentrated on only one of them – this is the hinge that will creak. The cylinder of that hinge presses down on its metal base with the weight of the entire door, whose mass is M.

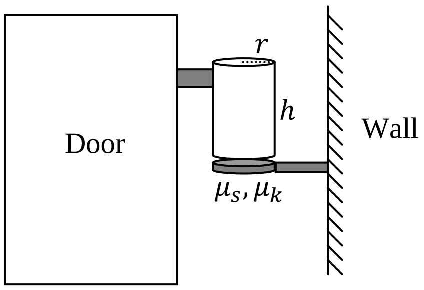

Figure 2: Schematic drawing of a door

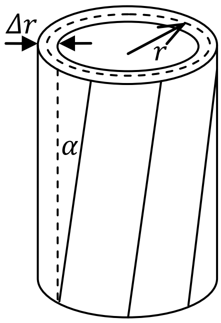

Figure 3: The twisted hinge cylinder

The cylinder is not a perfectly rigid body – it can twist tangentially without changing its overall form, so that vertical line segments become tilted with a small angle α; see Figure 3. The elastic force on a small area element dS of the base due to this deformation is given by:

$$dF = G\alpha dS,$$

where G is a material property known as the shear modulus. Use the values $r = 5\text{mm}$, $h = 3\text{cm}$, $\Delta r = 1\text{mm}$, $\mu_s = 0.75$, $\mu_k = 0.55$, $G = 8 \cdot 10^{10}\text{Pa}$, $M = 30\text{kg}$, $g = 9.8\text{m/s}^2$. You may use the approximation $\Delta r \ll r$.

f. (1 pt.) We start rotating the door very slowly from equilibrium (zero torque). For small rotation angles, obtain an expression for the torsion coefficient $\kappa = \tau/\theta$, where τ is the torque which must be applied to rotate the door by an angle θ.

g. (1.5 pts.) At very low angular velocity, when a transition from stick to slip occurs, a sound pulse is emitted. Find the angular velocity Ω of the door for which the frequency of these pulses enters the audible range at $f = 20\text{Hz}$.

Assume that the frequency f_0 of pure-slip oscillations in the hinge is much higher: $f_0 \gg f$. Provide an expression and a numerical result.

Theoretical Question 3: Birthday Balloon

The picture shows a long rubber balloon, the kind that is popular at birthday parties. A partially inflated balloon usually splits into two domains of different radii. In this question, we consider a simplified model to help us understand this phenomenon.

Figure 1: A partially inflated birthday balloon.

Consider a balloon with the shape of a long homogeneous cylinder (except for the ends), with a mouthpiece through which the balloon can be inflated. All processes will be considered isothermal at room temperature. At all times, the pressure P inside the balloon exceeds the atmospheric pressure P_0 by a small fraction, so the air may be treated as an incompressible fluid. Gravity and the balloon's weight may also be neglected. The inflation is slow and quasistatic. In parts (a)-(d), the balloon is inflated uniformly throughout its length. We denote by r_0 and L_0 the radius and length of the balloon before it was inflated.

a. (1.8 pts.) The balloon is held by the mouthpiece, while its other parts hang freely. Find the ratio σ_L/σ_t between the longitudinal surface tension σ_L (in the direction parallel to the balloon's axis) and the transverse surface tension σ_t (in the direction tangent to the balloon's circular cross-section).
The surface tension of a rubber film is the force that adjacent parts exert on each other, per unit length of the boundary.

Hooke's Law is a linear approximation of real-world elasticity for small tensions. Assume that the balloon's length remains constant at L_0, while the surface tension σ_t depends linearly on the inflation ratio r/r_0:

$$\sigma_t = k\left(\frac{r}{r_0} - 1\right) \qquad (1)$$

b. (1 pt.) With these assumptions, obtain an expression for the dependence of the pressure P inside the balloon on the balloon's volume V. Sketch a plot of $P - P_0$ as a function of V. What is the maximal inflation pressure P_{max} resulting from Hooke's elasticity approximation?

In reality, because the inflation ratio r/r_0 is large (in Figure 1, typical values of about 5 can be observed), one must consider non-linear behavior of the rubber and changes in the balloon's length. These effects allow higher inflation pressures than predicted by part (b). In a typical balloon, the graph of $\sigma_t(r)$ is composed of three pieces:

1. For small inflation ratios, $\sigma_t(r)$ grows in a Hooke-like manner.
2. At $r - r_0 \sim r_0$, the balloon's length L begins to increase, and $\sigma_t(r)$ reaches a long plateau where it grows very slowly.
3. At some large inflation ratio, the rubber starts strongly resisting any further stretch, which leads to a sharp rise in $\sigma_t(r)$.

This behavior is depicted in Figure 2.

c. (1.3 pts.) Sketch a qualitative plot of the pressure difference $P - P_0$ as a function of V for a uniformly inflated balloon that behaves according to Figure 2. Indicate any local extremum points on your plot. Indicate also the points corresponding to $r = 1\text{cm}$ and $r = 2.5\text{cm}$. Find the values of $P - P_0$ at these two points with 10% accuracy.

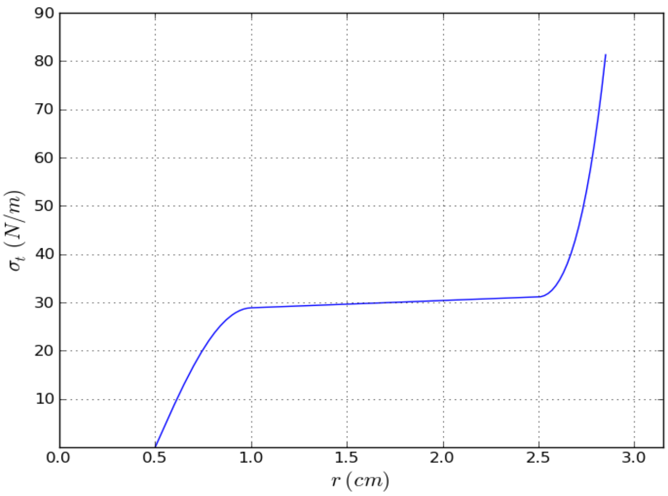

Figure 2: $\sigma_t(r)$ for a realistic party balloon.

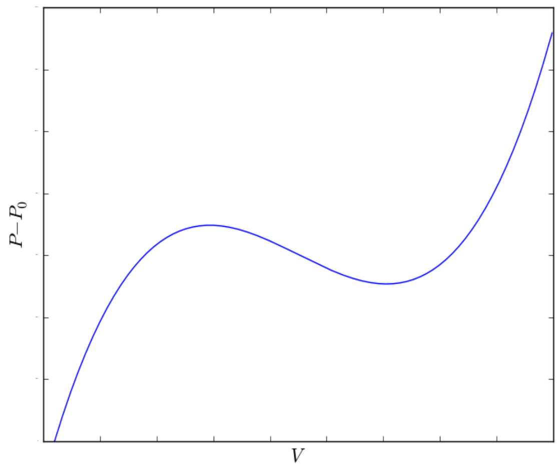

Figure 3: A plot of equation (2).

To explore the consequences of the behavior you found in part (c), we approximate $P(V)$ for a uniformly inflated balloon with a cubic function:

$$P - P_0 = a((V-u)^3 - b(V-u) + c) \qquad (2)$$

where a, b, c and u are positive constants. Assume that the volume u is larger than the balloon's uninflated volume V_0, and c is large enough so that the function (2) is positive in the entire physical range $V > V_0$. See Figure (3).

The balloon is attached to a large air reservoir maintained at a controllable pressure P. It may happen that some values of P are consistent with more than one value of the volume V. If the balloon suffers occasional perturbations (such as local stretching by external forces) while held at such inflation pressure, it may jump to a state of different volume. This will happen when it becomes energetically favorable for the entire system, consisting of the balloon, the atmosphere and the machinery maintaining the pressure P. If the pressure is slowly increased from P_0, and sufficient perturbations exist at every step, this explosive volume jump will happen at a certain pressure P_c where the energy required to move between the two states is zero. Above this pressure, going from the smaller volume to the larger volume branch releases energy, and vice versa. This type of discontinuity is often found in nature, and is sometimes referred to as a "phase transition".

d. (2.3 pts.) By considering equation (2), obtain the value of P_c, the volume V_1 of the balloon before the jump, and the volume V_2 after the jump. Express your answers using a, b, c and u.

A more realistic inflating agent, such as a birthday boy, is unable to supply enough air for the instantaneous volume change described above. Instead, air is pumped gradually into the balloon, effectively controlling the balloon's volume rather than the pressure. In this case, a new type of behavior becomes possible. If it helps to minimize the total energy of the system, the balloon will split (given sufficient perturbations) into two cylindrical domains of different radii, whose lengths will gradually change. The splitting boundary itself requires energy, which you may neglect. We shall also neglect the length of the boundary layer (these assumptions are valid for a very long balloon.)

e. (1 pt.) Sketch a qualitative graph of the pressure difference $P - P_0$ as a function of V, taking the split into account. Indicate on your axes the pressure $P_c - P_0$ and the volumes V_1 and V_2.

f. (1.4 pts.) The balloon is in the volume range that supports two coexisting domains. Find the length L_{thin} of the thinner domain as a function of the total air volume V. Express your answer in terms of V_1, V_2 and the radius r_1 of the thinner domain.

g. (1.2 pts.) The balloon is in the volume range that supports two coexisting domains. Find the latent work $\Delta W / \Delta L_{thin}$ that must be performed on the balloon to convert a unit length of the thin domain into the thick domain. Express your answer in terms of P_c, V_1, V_2 and the radius r_1 of the thinner domain.