The Drag on a Falling Magnet

A clear and detailed discussion on eddy currents was first provided by the British physicist Sir James H. Jeans (1877-1946) in his celebrated book *The mathematical theory of electricity and magnetism* (1925). The present problem is based on electricity and magnetism.

James H. Jeans
(1877-1946)

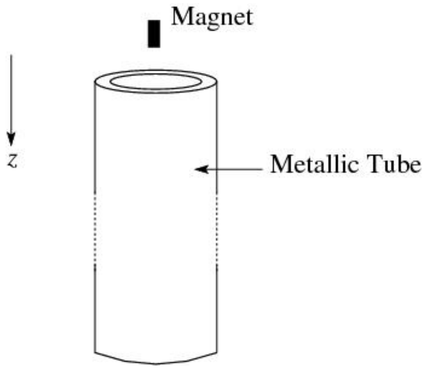


Figure 1

A small size magnet with dipole moment of magnitude p and mass m is dropped through a very long vertically held non-magnetic metallic tube as shown in Fig. (1) (figure is not to scale). In general the fall is governed by

$$m\ddot{z} = mg - k\dot{z} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Here g is the acceleration due to gravity. Note that the damping parameter k is due to the generation of eddy currents in the tube.

I.1 Obtain the terminal velocity (v_T) of the magnet. **[0.5 point]**

I.2 Obtain $z(t)$, i.e. position of the magnet at time t. Take $v(t=0)=0$ and $z(t=0)=0$. **[1.0 point]**

We shall attempt to understand the dynamics of the fall. In order to do this we consider in part **(I.3)** – part **(I.8)** a simplified problem of the magnet falling axially towards a fixed non-magnetic metallic ring of radius a, resistance R and inductance L as shown in Fig. (2). In this problem, we shall ignore radiation effects.

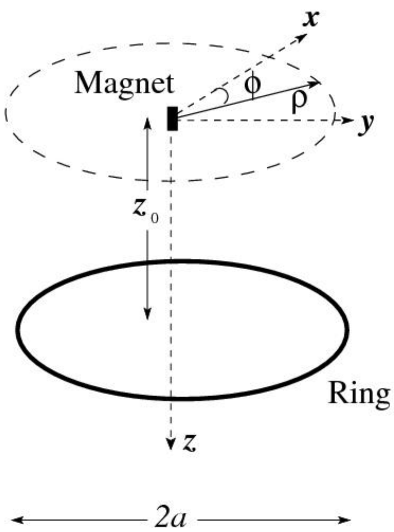


Figure 2

In our case it is convenient to change the reference coordinates to a set of cylindrical ones (ρ, φ, z) as shown in Fig. (2) where z-axis is the ring axis, the magnet is initially at rest at the origin and the center of the ring is at distance z_0 from the origin. Cartesian axes (x, y, z) are also shown in the figure. The magnet has dipole moment $\vec{p}$ in the positive z direction ($\vec{p} = p\hat{k}$) where $\hat{k}$ is unit vector in z direction. We will assume that during the fall, magnetic moment remains in the same direction. The axial component (B_z) and radial component (B_ρ) of the magnetic field at an arbitrary point (ρ, φ, z) when the magnet is at the origin are given by

$$B_z = \frac{\mu_0}{4\pi}\frac{p}{(\rho^2+z^2)^{\frac{3}{2}}}\left[\frac{3z^2}{\rho^2+z^2}-1\right]$$
$$B_\rho = \frac{\mu_0}{4\pi}\frac{3pz\rho}{(\rho^2+z^2)^{5/2}}$$

where μ_0 is the permeability of free space.

I.3 Let the instantaneous speed of the magnet be v. Obtain the magnitude of the induced emf (e_i) in the ring. **[1.5 points]**

I.4 This emf will give rise to an induced current (i) in the ring. Obtain the magnitude of the instantaneous electromagnetic force (f_{em}) on the ring in terms of i. **[1.0 point]**

I.5 What is the magnitude of the force on the magnet due to this ring? **[0.5 point]**

I.6 Express the emf in the ring in terms of L, R and i. Do not solve for i. **[0.5 point]**

I.7 As the magnet falls it loses gravitational potential energy. Identify the three main forms of energy into which the gravitational potential energy is converted and write down the expressions you would use to calculate each of the three contributions. **[1.0 point]**

I.8 Does the magnetic field of the magnet do any work in this process? Tick in the appropriate box. **[0.5 point]**

Next we will estimate the damping parameter k due to the pipe (see Eq. (1)). Take an infinitely long pipe with radius a, small thickness w, and electrical conductivity σ. For this and later part, we take inductance of the pipe to be negligible. It would help if you considered the pipe to be made of many rings each of height $\Delta z'$, radius a, small thickness w and electrical conductivity σ (see Fig. (3)). For simplicity, the two ends of the pipe are at $z = -\infty$ and at $z = \infty$, respectively.

I.9 Obtain the resistance of an individual ring. **[0.5 point]**

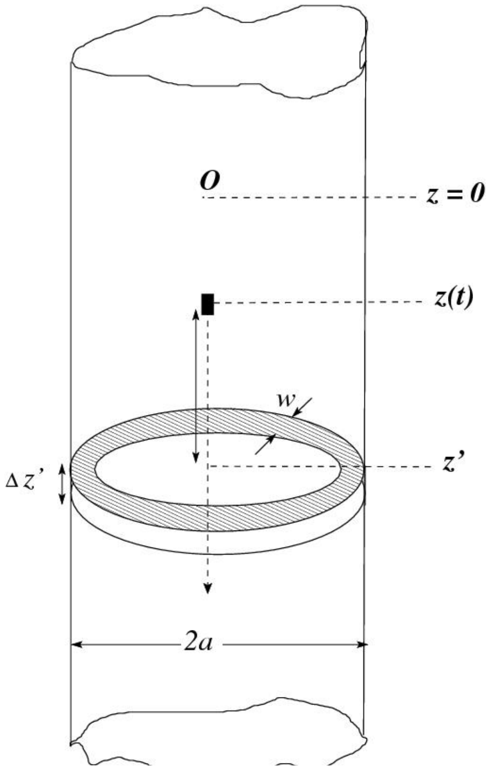


Figure 3

I.10 Obtain the damping parameter k due to the entire pipe in terms of p, σ and geometrical parameters of the ring. Since each ring is very thin, you may take magnetic field to be constant over the thickness of the ring and equal to $B_\rho(\rho = a)$. Assume that at an instant t, the magnet has a coordinate $z(t)$ with an instantaneous speed $\dot{z}$. You should leave your answer in terms of a dimensionless integral I, involving a dimensionless variable $u = (z - z')/a$. **[2.0 points]**

I.11 Assume that the damping constant k depends on the following

$$k = f(\mu_0, p, R_0, a)$$

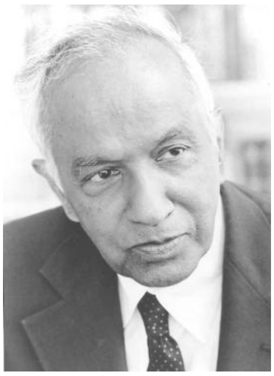
where R_0 is the effective resistance of a long pipe. Use dimensional analysis to obtain an expression for k. Take the dimensionless constant to be unity. **[1.0 point]**

The following integral may be useful:

$$\int \frac{u\,du}{(u^2 + a^2)^n} = \frac{1}{2}\frac{(a^2 + u^2)^{1-n}}{1 - n} + \text{Constant} \qquad (\text{n} > 1)$$

Chandrasekhar Limit

In a famous work carried out in 1930, the Indian Physicist Prof Subrahmanyan Chandrasekhar (1910-1995) studied the stability of stars. The problem will help you to construct a simplified version of his analysis.



S. Chandrasekhar
(1910-1995)

You may find the following symbols and values useful.

Speed of light in vacuum	$c = 3.00 \times 10^{8} \mathrm{m.s^{-1}}$
Planck's constant	$h = 6.63 \times 10^{-34} \mathrm{J.s}$
Universal constant of Gravitation	$G = 6.67 \times 10^{-11} \mathrm{N.m^2.kg^{-2}}$
Rest mass of electron	$m_e = 9.11 \times 10^{-31} \mathrm{kg}$
Rest mass of proton	$m_p = 1.67 \times 10^{-27} \mathrm{kg}$

II.1. Consider a spherical star of uniform density, radius R and mass M. Derive an expression for its gravitational potential energy (E_G) due to its own gravitational field (gravitational self energy). **[1.0 point]**

II.2. We assume that the star is made up of only hydrogen and that all the hydrogen is in ionized form. We consider the situation when the star's energy production due to nuclear fusion has stopped. Electrons obey the Pauli exclusion principle and their total energy can be computed using quantum statistics. You may take this total electronic energy (ignoring the protonic energy) to be

$$E_e = \frac{\hbar^2 \pi^3}{10 m_e 4^{2/3}} \left(\frac{3}{\pi}\right)^{7/3} \frac{{N_e}^{5/3}}{R^2}$$

where N_e is the total number of electrons and $\hbar = h/2\pi$. Obtain the equilibrium condition of the star relating its radius (R_{wd}) to its mass. This radius is called the 'White Dwarf' radius. **[2.0 points]**

II.3. Numerically evaluate R_{wd} given that mass of the star is the same as the solar mass ($M_S = 2.00 \times 10^{30} \mathrm{kg}$). **[1.5 points]**

II.4. Assuming that the electron distribution is homogeneous, obtain an order of magnitude estimation of the average separation (r_{sep}) between electrons if the radius of the star is R_{wd} as obtained in part (II.3). **[1.0 point]**

II.5. Let us estimate the speed of electrons. For this purpose, assume each electron to form a standing wave in a one-dimensional box of length r_{sep}. Estimate the speed of electron (v) in the lowest energy state using de-Broglie hypothesis **[1.0 point]**

II.6. Consider now a modification of the analysis in part (II.2). If we take electrons in the ultrarelativistic limit ($E = pc$), a similar analysis yields

$$E_e^{rel} = \frac{\pi^2}{4^{4/3}}\left(\frac{3}{\pi}\right)^{5/3}\frac{\hbar c}{R}{N_e}^{4/3}$$

Obtain the expression for the mass for which, the star can be in equilibrium in terms of the constants provided at the beginning of the question. We call this the critical mass (M_c). **[1.5 points]**

II.7. If the mass M of the star is greater than the critical mass M_c obtained in part (II.6), state whether the star will expand or contract. Tick in appropriate box.

[0.5 point]

II.8. Calculate a numerical estimate of this critical mass in units of solar mass (M_S).

(Note: Your answer may differ from Chandrasekhar's famous result because of the approximations made in this analysis)

[1.5 points]

Pancharatnam Phase

This problem deals with the two beam phenomena associated with light, its interference, polarization and superposition. The particular context of the problem was studied by the Indian physicist S. Pancharatnam (1934–1969).

S. Pancharatnam
(1934–1969)

Consider the experimental set up as shown in Fig. (1). Two coherent monochromatic light beams (marked as beam 1 and 2), travelling in the z direction, are incident on two narrow slits and separated by a distance d ($S_1S_2 = d$). After passing through the slits the two beams interfere and the pattern is observed on the screen S. The distance between the slits and the screen is D and $D \gg d$. Assume that the width of each slit S_1 and S_2 is much smaller than the wavelength of light.

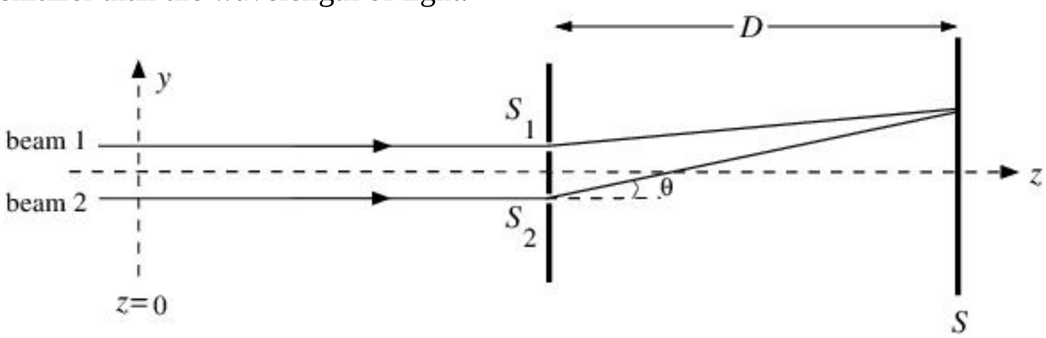


Figure 1

III.1. Let the beams 1 and 2 be linearly polarized at $z = 0$. The corresponding electric field vectors are given by

$$\vec{E}_1 = \hat{\imath} E_0 \cos(\omega t) \qquad \text{.....................(1a)}$$

$$\vec{E}_2 = \hat{\imath} E_0 \cos(\omega t) \qquad \text{.....................(1b)}$$

where $\hat{\imath}$ is the unit vector along the x-axis, ω is angular frequency of light and E_0 is the amplitude. Find the expression for the intensity of the light $I(\theta)$, that will be observed on the screen where θ is the angle shown in Fig. (1). Express your answer in terms of θ, d, E_0, c and ω where c is the speed of light. Also, note that the intensity is proportional to the time average of the square of the electric field. Here you make take the proportionality constant to be β. You may ignore the attenuation in the magnitude of the electric fields with distance from the slits to any point on the screen.

[1.0 point]

III.2. A perfectly transparent glass slab of thickness w and refractive index μ is introduced in the path of beam 1 before the slits. Find the expression for the intensity of the light $I(\theta)$ that will be observed on the screen. Express your answer in terms of $\theta, d, E_0, c, \omega, \mu$ and w.

[1.0 point]

III.3. An optical device (known as quarter wave plate (QWP)) is introduced in the path of beam 1, before the slits, replacing the glass slab. This device changes the polarization of the beam from the linear polarization state

$$\vec{E}_1 = \hat{\imath} E_0 \cos(\omega t)$$

to a circular polarization state which is given by

$$\vec{E}_1 = \frac{1}{\sqrt{2}}[\hat{\imath} E_0 \cos(\omega t) + \hat{\jmath} E_0 \sin(\omega t)] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $\hat{\jmath}$ is the unit vector along the y-axis.
Assume that the device does not introduce any additional path difference and that it is perfectly transparent. Note that the tip of the electric field vector traces a circle as time elapses and hence, the beam is said to be circularly polarized. We assume that the angle θ is small enough so that intensity from slit one does not depend on the angle θ even for $\hat{\jmath}$ polarization.

III.3.a. Find the expression for the intensity $I(\theta)$ of the light that will be observed on the screen. Express your answer in terms of θ, d, E_0, c and ω.
III.3.b. What is the maximum intensity (I_{max})?
III.3.c. What is the minimum intensity (I_{min})?

[2.0 points]

III.4.

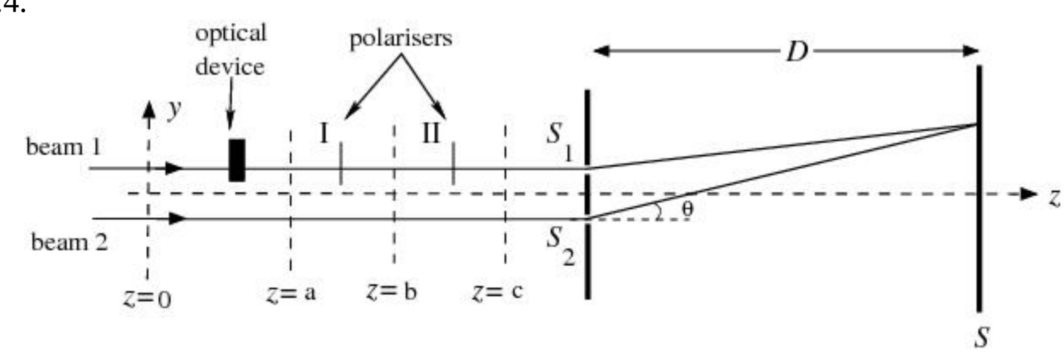


Figure 2

Now, consider the experimental setup (see Fig. (2)) in which the beam 1 is subjected to

- the device (QWP) described in part 3 and,
- a linear polarizer (marked as I), between $z = a$ and $z = b$ which allows only the component of the electric field parallel to an axis ($\hat{\imath}'$) to pass through. The

unit vector $\hat{\imath}'$ is defined as

$$\hat{\imath}' = \hat{\imath}\cos\gamma + \hat{\jmath}\sin\gamma$$

and,

- another linear polarizer (marked as II) between $z = b$ and $z = c$ which polarizes the beam back to $\hat{\imath}$ direction.

Thus the beam 1 is back to its original state of polarization. Assume that the polarizers do not introduce any path difference and are perfectly transparent.

III.4.a. Write down the expression for the electric field of beam 1 after the first polarizer at $z = b$ $[\vec{E}_1(z = b)]$.

III.4.b. Write down the expression for the electric field of beam 1 after the second polarizer at $z = c$ $[\vec{E}_1(z = c)]$.

III.4.c. What is the phase difference (α) between the two beams at the slits?

[2.0 points]

The most general type of polarization is elliptical polarization. A convenient way of expressing elliptical polarization is to consider it as a superposition of two orthogonal linearly polarized components i.e.

$$\vec{E} = \hat{\imath}' E_0 \cos e \cos(\omega t) + \hat{\jmath}' E_0 \sin e \sin(\omega t) \quad \text{..................(3)}$$

where $\hat{\imath}'$ and $\hat{\jmath}'$ and this state of polarization are depicted in Fig. 3.

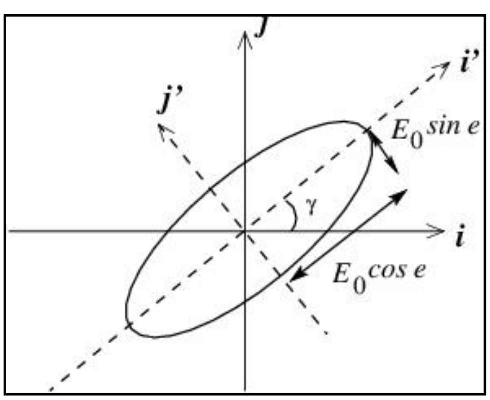


Figure 3

The tip of the electric field vector traces an ellipse as time elapses. Here e represents the ellipticity and is given by

$$\tan e = \frac{\text{Semi-minor axis of the ellipse}}{\text{Semi-major axis of the ellipse}}$$

Linear polarization (Eqs. (1)) and circular polarization (Eq. (2)) are special cases of elliptical polarization (Eq. (3)). The two parameters $\gamma(\in[0,\pi])$ and $e(\in[-\pi/4,\pi/4])$ completely describe the state of polarization.

The polarization state can also be represented by a point on a sphere of unit radius called the Poincare sphere. The polarization of the beam described in Eq. (3) is represented by a point P on the Poincare sphere (see Fig. 4), then latitude $\angle PCD = 2e$ and longitude $\angle ACD = 2\gamma$. Here C is the center.

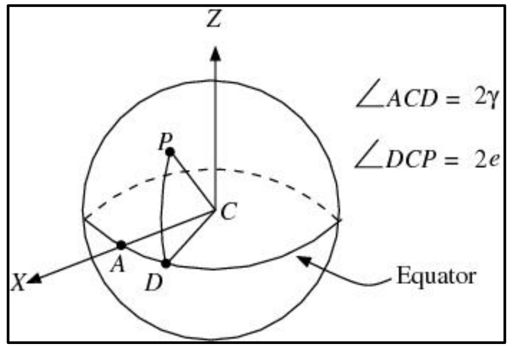


Figure 4

III.5. Consider a point on the equator of the Poincare sphere.

III.5.a. Write down the electric field ($\vec{E}_{\mathrm{Eq}}$) corresponding to this point.

III.5.b. What is its state of polarization?

[0.5 point]

III.6. Consider a point at the north pole of the Poincare sphere.

III.6.a Write down the electric field ($\vec{E}_{\mathrm{NP}}$) corresponding to this point.

III.6.b What is its state of polarization?

[0.5 point]

III.7. Now, consider the three polarization states of beam 1 as given in part 4. Let the initial polarization (at $z = 0$) be represented by a point A_1 on the Poincare sphere; after the optical device, let the state (at $z = a$) be represented by point A_2 and after the first polarizer (say, at $z = b$), the state be represented by point A_3. At $z = c$, the polarization returns to its original state which is represented by A_1. Locate these points ($A_1, A_2,$ and A_3) on the Poincare sphere.

[1.5 points]

III.8. If these three points ($A_1, A_2,$ and A_3 from the part (III.7)) are joined by great circles on the sphere, a triangle on the surface of the sphere is obtained (Note: A great circle is a circle on the sphere whose center coincides with the center of the sphere). The phase difference α obtained in part 4 and the area S of the curved surface enclosed by the triangle are related to each other. Relate S to α.

This relationship is general and was obtained by Pancharatnam and the phase difference is called the *Pancharatnam phase*. **[1.5 points]**