I.1. Equation of motion for the magnet is

$$m\ddot{z} = mg - k\dot{z} \tag{1}$$

For terminal velocity $\ddot{z} = 0$ $\boxed{0.2 \text{ mark}}$

which gives $v_T = \dot{z} = \dfrac{mg}{k}$ $\boxed{0.3 \text{ mark}}$

I.2. Rewriting Eq. (1)

$$\frac{dv}{dt} = g - \frac{k}{m}v(t)$$

Given that $v(t=0) = 0; z(t=0) = 0$ which yields

$$v(t) = \frac{mg}{k}(1 - e^{-kt/m}) = \frac{dz}{dt} \qquad \boxed{0.5 \text{ mark}}$$

$$\int_0^z dz = \int_0^t \frac{mg}{k}(1 - e^{-kt/m})dt$$

$$z(t) = \frac{mg}{k}\left[t + \frac{m}{k}(e^{-kt/m} - 1)\right] \qquad \boxed{0.5 \text{ mark}}$$

I.3. **Method - I** : Because of the relative speed v between the magnet and the ring, in the field $\vec{B} = B_z\hat{k} + B_\rho\hat{\rho}$ of the magnet, the induced emf is given by

$$e_i = \int (\vec{v} \times \vec{B}).d\vec{l} \qquad \boxed{0.8 \text{ mark}}$$

$$e_i = vB_a 2\pi a \qquad \boxed{0.4 \text{ mark}}$$

where
$$B_a = \frac{\mu_0}{4\pi}\frac{3pa(z_0 - z)}{[a^2 + (z_0 - z)^2]^{5/2}} \tag{2}$$

$\boxed{0.3 \text{ mark}}$

Method - II :Magnetic flux (ϕ) through the ring is

$$\phi = \int_0^a B_z 2\pi\rho d\rho \qquad \boxed{0.2 \text{ mark}}$$

$$= 2\pi\int_0^a \frac{\mu_0}{4\pi}\frac{\rho p}{(\rho^2 + (z_0 - z)^2)^{3/2}}\left[\frac{3(z_0 - z)^2}{\rho^2 + (z_0 - z)^2} - 1\right]d\rho \qquad \boxed{0.2 \text{ mark}}$$

$$\phi = \frac{\mu_0 p a^2}{2(a^2 + (z_0 - z)^2)^{3/2}} \qquad \boxed{0.3 \text{ mark}}$$

Induced emf $e_i = \dfrac{-d\phi}{dt} = -v\dfrac{d\phi}{dz}$ $\boxed{0.4 \text{ mark}}$

which gives $e_i = \dfrac{\mu_0 3pa^2 v(z_0 - z)}{2[a^2 + (z_0 - z)^2]^{5/2}}$ $\boxed{0.4 \text{ mark}}$

I.4. B_z component will cause a radially outward force on the ring and by symmetry this yields a null force. 0.4 mark

Only B_ρ will contribute to

$$\vec{df}_{em} = i(\vec{dl} \times \vec{B})$$
$$\left|\vec{f}_{em}\right| = i2\pi a B_a$$ 0.6 mark

where B_a is given by Eq. (2).

I.5. By Newton's third law, equal and opposite force will be exerted by the ring on the magnet. Hence the magnitude of the force on the magnet by the ring is f_{em}. 0.5 mark

I.6. $e_i = L\dfrac{di}{dt} + iR$ 0.5 mark

I.7. Potential energy is converted to three parts:

(a) $mv^2/2$ (kinetic energy) 0.3 mark

(b) $Li^2/2$ (magnetic energy) 0.3 mark

(c) $i^2R\Delta t$ (Joule loss due to the current in time Δt). 0.4 mark

I.8. The magnetic field does no work in the process.

Yes	
No	✓

0.5 mark

I.9. Resistance of the ring

$$\Delta R = \frac{2\pi a}{\sigma w \Delta z'}$$ 0.5 mark

I.10. Now, the net force on the magnet, due to one ring at z' is given by

$$f_{em} = (2\pi a) i B'_a$$

where

$$B'_a = \frac{\mu_0}{4\pi} \frac{3pa(z'-z)}{(a^2 + (z'-z)^2)^{5/2}}$$ 0.3 mark

and i is the induced current in the ring which is given by

$$i = \frac{e_i}{\Delta R} = \frac{\sigma w e_i}{2\pi a} \Delta z'$$ 0.5 mark

Then the net force on the magnet due to the entire pipe is given by

$$F = \int_{-\infty}^{\infty} f_{em} = \int_{-\infty}^{\infty} B'^2_a (2\pi a) w \sigma dz'.\dot{z}$$ 0.2 mark

Since the pipe is very long the limits of integration can be taken as $-\infty$ and ∞. Substituting B'_a, we get

$$F = \left(\frac{\mu_0}{4\pi}\right)^2 18p^2a^3\pi w\sigma\dot{z}\int_{-\infty}^{\infty}\frac{(z'-z)^2}{((z'-z)^2+a^2)^5}dz' \qquad \boxed{0.5 \text{ mark}}$$

Let $u = (z'-z)/a$. Finally,

$$F = \left(\frac{\mu_0}{4\pi}\right)^2 \frac{18p^2\pi\sigma w\dot{z}}{a^4}\int_{-\infty}^{\infty}\frac{u^2}{(1+u^2)^5}du$$

Thus damping parameter

$$k = \left(\frac{\mu_0}{4\pi}\right)^2 \frac{18p^2\pi\sigma w}{a^4}\int_{-\infty}^{\infty}\frac{u^2}{(1+u^2)^5}du \qquad \boxed{0.5 \text{ mark}}$$

I.11. Given that,

$$k = f(\mu_0, p, R_0, a)$$

Dimensions of various parameters involved are

$$[\mu_0] = I^{-2}MLT^{-2} \qquad \boxed{0.2 \text{ mark}}$$
$$[p] = IL^2 \qquad \boxed{0.1 \text{ mark}}$$
$$[R_0] = I^{-2}ML^2T^{-3} \qquad \boxed{0.2 \text{ mark}}$$
$$[a] = L$$
$$[k] == MT^{-1}$$

which gives

$$k = \frac{p^2\mu_o^2}{a^4R_0} \qquad \boxed{0.5 \text{ mark}}$$

II.1. Consider a shell of width dr which is at the distance r from the centre of the star. Let ρ be the density of star. Gravitational Potential energy of shell is

$$dE_G = -G\frac{(4\pi r^3\rho/3)(4\pi r^2 dr\rho)}{r} \qquad \boxed{0.5 \text{ mark}}$$

$$E_G = \int_0^R dE_G = -\frac{3}{5}\frac{GM^2}{R} \qquad \boxed{0.3 \text{ mark}}$$

It is negative, i.e. gravitational force is radially inward. $\boxed{0.2 \text{ mark}}$

II.2. Total energy of the star $E = E_G + E_e$. $\boxed{0.2 \text{ mark}}$
At equilibrium, at $R = R_{wd}$

$$\frac{dE}{dR} = 0 \qquad \boxed{0.8 \text{ mark}}$$

$$\frac{dE_G}{dR} = -\frac{dE_e}{dR}$$

$$\frac{3}{5}\frac{GM^2}{R_{wd}^2} = \frac{\hbar^2\pi^3}{10m_e 4^{2/3}}\left(\frac{3}{\pi}\right)^{7/3}\frac{2N_e^{5/3}}{R_{wd}^3}$$

$$R_{wd} = \frac{\hbar^2\pi^3}{6Gm_e 4^{2/3}}\left(\frac{3}{\pi}\right)^{7/3}\frac{2N_e^{5/3}}{M^2} \qquad \boxed{1.0 \text{ marks}}$$

II.3. Since all the hydrogen is ionized, the number of protons $(N_p) = N_e$. Also $m_p \gg m_e$. Hence

$$N_e = N_p \approx \frac{M}{m_p} \qquad \boxed{0.5 \text{ mark}}$$

$$R_{wd} = \frac{\hbar^2\pi^3}{6Gm_e 4^{2/3}}\left(\frac{3}{\pi}\right)^{7/3}\frac{2}{M^{1/3}m_p^{5/3}}$$

$$R_{wd} = 2.28 \times 10^4 \text{ km} \qquad \boxed{1.0 \text{ mark}}$$

II.4. If r_{sep} is average separation between electrons then

$$N_e \times \frac{4}{3}\pi r_{sep}^3 \approx \frac{4}{3}\pi R_{wd}^3 \qquad \boxed{0.2 \text{ mark}}$$

$$r_{sep}^3 = \frac{R_{wd}^3}{N_e} \approx R_{wd}^3\frac{m_p}{M} \qquad \boxed{0.2 \text{ mark}}$$

$$r_{sep} = 2.13 \times 10^{-12} \text{ m} \qquad \boxed{0.6 \text{ mark}}$$

II.5. For a particle confined in a box of length r_{sep}, its de Broglie wavelength λ_{dB} for the ground state can be written as

$$\lambda_{dB} = 2r_{sep} \qquad \boxed{0.2 \text{ mark}}$$

$$\text{and momentum } p = \frac{h}{\lambda_{dB}} \qquad \boxed{0.3 \text{ mark}}$$

$$v \approx \frac{h}{2m_e r_{sep}} \qquad \boxed{0.2 \text{ mark}}$$

$$= 1.08 \times 10^8 \text{ m.s}^{-1} \qquad \boxed{0.3 \text{ mark}}$$

Correction: If one takes relativistic momentum

$$\lambda_{dB} = 2r_{sep} \qquad \boxed{0.2 \text{ mark}}$$

$$p = \frac{h}{\lambda_{dB}} = \frac{h}{2r_{sep}} = \frac{m_e v}{\sqrt{1 - v^2/c^2}} \qquad \boxed{0.3 \text{ mark}}$$

$$v = \frac{h}{\sqrt{4m_e^2 r^2 + \frac{h^2}{c^2}}} \qquad \boxed{0.2 \text{ mark}}$$

$$= 1.06 \times 10^8 \text{ m.s}^{-1} \qquad \boxed{0.3 \text{ mark}}$$

II.6. Similar to part II.2, at equilibrium

$$\frac{dE_G}{dR} = -\frac{dE_e^{rel}}{dR} \qquad \boxed{0.3 \text{ mark}}$$

$$\frac{3}{5}\frac{GM^2}{R^2} = \frac{\pi^2}{4^{4/3}}\left(\frac{3}{\pi}\right)^{5/3}\frac{\hbar c}{R^2}N_e^{4/3} \qquad \boxed{0.4 \text{ mark}}$$

For critical mass

$$M_c = \frac{3(5^3\pi)^{1/2}}{16m_p^2}\left(\frac{\hbar c}{G}\right)^{3/2} \qquad \boxed{0.8 \text{ mark}}$$

Alternatively: Since the total energy

$$E_G + E_e^{rel} = \left(-\frac{3}{5}GM^2 + \frac{\pi^2}{4^{2/3}}\left(\frac{3}{\pi}\right)^{5/3}\hbar c N_e^{4/3}\right)\frac{1}{R}$$

must be minimized for equilibrium, one can argue that if coefficient of $1/R$ is positive then star would collapse otherwise it would expand.

II.7. For $M > M_c$

Expand	
Contract	✓

$\boxed{0.5 \text{ mark}}$

II.8.

$$M_c = 1.36 \times 10^{31} \text{ kg} \qquad \boxed{1.0 \text{ mark}}$$

$$= 6.8\ M_S \qquad \boxed{0.5 \text{ mark}}$$

III.1. **[1.0 mark]** Let the phase difference between two rays making an angle θ with z direction be δ. Clearly

$$\delta = \omega \Delta t = \omega d \sin\theta / c \tag{1}$$

Then the intensity is given by

$$I(\theta) = \beta \overline{|\mathbf{E}_1 + \mathbf{E}_2|^2}$$

Here

$$\begin{aligned} |\mathbf{E}_1 + \mathbf{E}_2|^2 &= |E_0 \cos(\omega t) + E_0 \cos(\omega t + \delta)|^2 \\ &= |E_0 \cos(\omega t)(1 + \cos\delta) - E_0 \sin(\omega t) \sin\delta|^2 \\ &= E_0^2 \left[\cos^2(\omega t)\left(1 + 2\cos\delta + \cos^2\delta\right) + \sin^2(\omega t)\sin^2\delta \right. \\ &\quad \left. -2\cos(\omega t)(1+\cos\delta)\sin(\omega t)\sin\delta\right] \end{aligned}$$

Since $\overline{\cos^2 \omega t} = \overline{\sin^2 \omega t} = 1/2$ and $\overline{\sin(\omega t)\cos(\omega t)} = 0$, we get the intesity to be

$$I(\theta) = \beta E_0^2 (1 + \cos\delta)$$

where δ is given by Eq. (1).

(a) Alternate:

$$\begin{aligned} |\mathbf{E}_1 + \mathbf{E}_2|^2 &= |E_0 \cos(\omega t) + E_0 \cos(\omega t + \delta)|^2 \\ &= |E_0 \cos(\omega t)(1 + \cos\delta) + E_0 \sin(\omega t)\sin\delta|^2 \\ &= |E_0 A \cos(\omega t - \phi)|^2 \end{aligned}$$

where

$$\begin{aligned} A^2 &= (1 + \cos\delta)^2 + \sin^2\delta \\ &= 2(1 + \cos\delta) \end{aligned}$$

Since $\overline{\cos^2(\omega t - \phi)} = 1/2$, we have

$$I(\theta) = \frac{\beta}{2} E_0^2 A^2 = \beta E_0^2 (1 + \cos\delta)$$

where δ is given by Eq. (1).

III.2. **[1.0 marks]** The beam 1 has travelled extra optical path $= (\mu - 1) w$. Thus the net phase difference between two beams when they emerge at the angle θ is

$$\delta = \frac{\omega}{c}\left(d \sin\theta - (\mu - 1) w\right)$$

and $I(\theta) = \beta E_0^2 (1 + \cos\delta)$.

III.3. **[2.0 marks]** The two beams have travelled exactly same paths upto the slits. Thus when they emerge at an angle θ, they have a net phase difference of $\delta = \omega d \sin\theta / c$. Then, notice

$$\begin{aligned}
|\mathbf{E}_1 + \mathbf{E}_2|^2 &= \left| \mathbf{i}\left[\frac{E_0}{\sqrt{2}}\cos(\omega t) + E_0\cos(\omega t + \delta)\right] + \mathbf{j}\left[\frac{E_0}{\sqrt{2}}\sin(\omega t)\right]\right|^2 \\
&= \left| E_0\cos(\omega t)\left(\frac{1}{\sqrt{2}} + \cos\delta\right) + E_0\sin(\omega t)\sin\delta\right|^2 + \frac{E_0^2}{2}\sin^2(\omega t) \\
&= E_0^2\left[\cos^2(\omega t)\left(\frac{1}{2} + \sqrt{2}\cos\delta + \cos^2\delta\right) + \sin^2(\omega t)\sin^2\delta \right. \\
&\quad \left. + 2\cos(\omega t)\left(\frac{1}{\sqrt{2}} + \cos\delta\right)\sin(\omega t)\sin\delta\right] + \frac{E_0^2}{2}\sin^2(\omega t)
\end{aligned}$$

Thus, after taking time averages, the intesity will be

$$\begin{aligned}
I(\theta) &= \beta E_0^2\left[\frac{1}{2}\left(\frac{1}{2} + \sqrt{2}\cos\delta + \cos^2\delta\right) + \frac{1}{2}\sin^2\delta\right] + \beta\frac{E_0^2}{4} \\
&= \beta E_0^2\left[1 + \frac{1}{\sqrt{2}}\cos\delta\right] \\
&= \beta E_0^2\left[1 + \frac{1}{\sqrt{2}}\cos\left(\frac{\omega d\sin\theta}{c}\right)\right]
\end{aligned}$$

The maximum value of the intensity is $\beta E_0^2\left[1 + \frac{1}{\sqrt{2}}\right]$.

The minimum value of the intensity is $\beta E_0^2\left[1 - \frac{1}{\sqrt{2}}\right]$.

III.4. **[2.0 marks]** The electric field at $z = b$ is given by

$$\begin{aligned}
\mathbf{E}_1(z = b) &= \left(\frac{E_0}{\sqrt{2}}\left(\cos(\omega t - kb)\,\mathbf{i}\cdot\mathbf{i}' + \sin(\omega t - kb)\,\mathbf{j}\cdot\mathbf{i}'\right)\right)\mathbf{i}' \\
&= \frac{1}{\sqrt{2}}\left[E_0\cos\gamma\cos(\omega t - kb) + E_0\sin\gamma\sin(\omega t - kb)\right]\mathbf{i}' \\
&= \frac{1}{\sqrt{2}}\left[E_0\cos(\omega t - kb - \gamma)\right]\mathbf{i}'
\end{aligned}$$

and at $z = c$

$$\begin{aligned}
\mathbf{E}_1(z = c) &= \frac{E_0}{\sqrt{2}}\left(\cos(\omega t - kc - \gamma)\,\mathbf{i}'\cdot\mathbf{i}\right)\mathbf{i} \\
&= \frac{1}{\sqrt{2}}\cos\gamma E_0\cos(\omega t - kc - \gamma)\,\mathbf{i}
\end{aligned}$$

Where as

$$\mathbf{E}_2\,(z = c) = E_0\cos(\omega t - kc)\,\mathbf{i}$$

So, the net phase difference between beam 1 and beam 2 is now

$$\text{Phase difference } \alpha = \gamma.$$

III.5. **[0.5 marks]**

(a) At equator $e = 0$. Then

$$\mathbf{E} = \mathbf{E}_{\mathrm{Eq}} = \mathbf{i}' E_0 \cos(\omega t).$$

(b) Clearly the beam is linearly polarised along $\mathbf{i}'$.

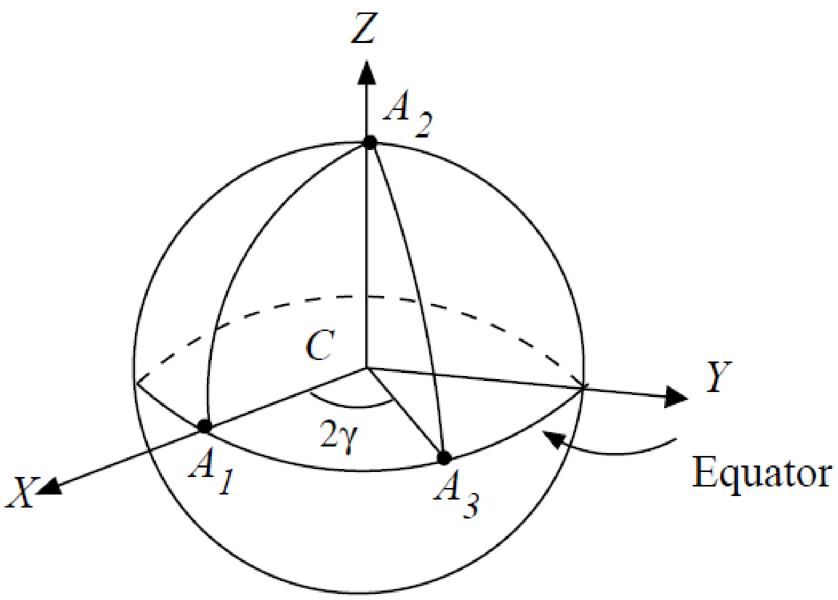
III.6. **[0.5 marks]**

(a) At north pole $e = \pi/4$ and γ can be taken to be 0. Then $\mathbf{i}' = \mathbf{i}$ and $\mathbf{j}' = \mathbf{j}$. The electric field

$$\mathbf{E}_{\mathrm{NP}} = \mathbf{i}\frac{E_0}{\sqrt{2}}\cos(\omega t) + \mathbf{j}\frac{E_0}{\sqrt{2}}\sin(\omega t),$$

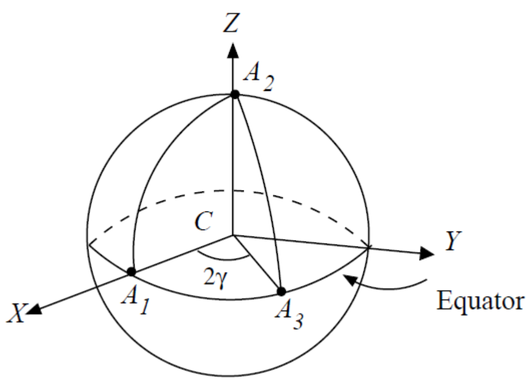
(b) Which represents circular polarisation.

III.7. **[1.5 marks]**



III.8. **[1.5 mark]** From figure it is clear that the area of the spherical triangle $A_1A_2A_3$ is $2\pi \times \left(\frac{2\gamma}{2\pi}\right) = 2\gamma$ and the phase difference was γ. Thus the phase difference is half the area of the spherical triangle $A_1A_2A_3$ on Poincare sphere.
Thus $S = 2\alpha$.
Here, the beam 1 passes through various states of polarization and returns to its original state. Though there has been no additional path difference, the beam has picked up a phase γ with respect to the beam 2. This is called as Pancharatnam phase.

QN	Solution	Marks
III.1	Step 1: Superposition $$\delta = \omega \Delta t = \frac{\omega d \sin\theta}{c}$$ $$\left|\mathbf{E}_1 + \mathbf{E}_2\right|^2 = E_0^2 \left[\cos^2(\omega t)\left(1 + 2\cos\delta + \cos^2\delta\right) + \sin^2(\omega t)\sin^2\delta - 2\cos(\omega t)(1+\cos\delta)\sin(\omega t)\sin\delta\right]$$	0.4 0.2
	Step 2: Time averaging: $\overline{\cos^2\omega t} = \overline{\sin^2\omega t} = 1/2$ and $\overline{\sin(\omega t)\cos(\omega t)} = 0$,	0.3
	Step 3: Intensity $$I(\theta) = \beta\overline{\left|\mathbf{E}_1 + \mathbf{E}_2\right|^2} = \beta E_0^2 (1 + \cos\delta)$$	0.1
III.2	The beam 1 has travelled extra optical path $= (\mu - 1)\, w$.	0.4
	Thus the net phase difference between two beams when they emerge at the angle θ, $$\delta = \frac{\omega}{c}\left(d\sin\theta - (\mu - 1)\, w\right)$$	0.2
	and $I(\theta) = \beta E_0^2 (1 + \cos\delta)$.	0.4
III.3	The two beams have travelled exactly same paths upto the slits. Thus when they emerge at an angle θ, they have a net phase difference of $\delta = \omega d \sin\theta / c$. Then, notice $$\left|\mathbf{E}_1 + \mathbf{E}_2\right|^2 = \left|\mathbf{i}\left[\frac{E_0}{\sqrt{2}}\cos(\omega t) + E_0\cos(\omega t + \delta)\right] + \mathbf{j}\left[\frac{E_0}{\sqrt{2}}\sin(\omega t)\right]\right|^2$$	0.70 for x component and 0.3 for y component
	$$\left|\mathbf{E}_1 + \mathbf{E}_2\right|^2 = E_0^2\left[\cos^2(\omega t)\left(\frac{1}{2} + \sqrt{2}\cos\delta + \cos^2\delta\right) + \sin^2(\omega t)\sin^2\delta + 2\cos(\omega t)\left(\frac{1}{\sqrt{2}} + \cos\delta\right)\sin(\omega t)\sin\delta\right] + \frac{E_0^2}{2}\cos^2(\omega t)$$	0.3
	Thus, after taking time averages, the intesity will be $$I(\theta) = \beta E_0^2\left[1 + \frac{1}{\sqrt{2}}\cos\left(\frac{\omega d\sin\theta}{c}\right)\right]$$	0.3

<table>
<tr><th>QN</th><th>Solution</th><th>Marks</th></tr>
<tr><td></td><td>The minimum value of the intensity is $\beta E_0^2\left[1-\frac{1}{\sqrt{2}}\right]$. Maximum value is $\beta E_0^2\left[1+\frac{1}{\sqrt{2}}\right]$</td><td>0.2+0.2</td></tr>
<tr><td>III.4</td><td>The electric field at $z=b$ is given b
$$\mathbf{E}_1(z=b) = \frac{1}{\sqrt{2}}\left[E_0\cos\left(\omega t-kb-\gamma\right)\right]\mathbf{i}'$$</td><td>magnitude 0.8 , direction 0.2</td></tr>
<tr><td></td><td>The electric field at $z=b$ is given by
$$\mathbf{E}_1(z=c) = \frac{1}{\sqrt{2}}\cos\gamma E_0\cos\left(\omega t-kc-\gamma\right)\mathbf{i}$$</td><td>magnitude 0.3, direction 0.2</td></tr>
<tr><td></td><td>Phase difference $\alpha=\gamma$.</td><td>0.5</td></tr>
<tr><td>III.5</td><td>At equator $e=0$. Then
$$\mathbf{E}=\mathbf{E}_{\mathrm{Eq}}=\mathbf{i}'E_0\cos\left(\omega t\right).$$
Clearly the beam is linearly polarised along $\mathbf{i}'$.</td><td>$\mathbf{E}$: 0.2 Linear Polarization: 0.3</td></tr>
<tr><td>III.6</td><td>At north pole $e=\pi/4$ and γ can be taken to be 0. Then $\mathbf{i}'=\mathbf{i}$ and $\mathbf{j}'=\mathbf{j}$. The electric field
$$\mathbf{E} = \mathbf{i}\frac{E_0}{\sqrt{2}}\cos\left(\omega t\right)+\mathbf{j}\frac{E_0}{\sqrt{2}}\sin\left(\omega t\right)$$
or
$$= \mathbf{i}\frac{E_0}{\sqrt{2}}\cos\left(\omega t+\gamma\right)+\mathbf{j}\frac{E_0}{\sqrt{2}}\sin\left(\omega t+\gamma\right)$$
which represents circular polarisation.</td><td>$\mathbf{E}$: 0.2 Circular Polarization: 0.3</td></tr>
<tr><td>III.7</td><td>Figure

</td><td>A1 on x axis 0.3, A2 on the north pole 0.5, A3 on equator 0.4, angle of 2γ 0.3</td></tr>
<tr><td>III.8</td><td>From figure it is clear that the area of the spherical triangle $A_1A_2A_3$ is $2\pi\times\left(\frac{2\gamma}{2\pi}\right)=2\gamma$</td><td>0.7</td></tr>
<tr><td></td><td>Thus the phase difference α is half the area S of the spherical triangle $A_1A_2A_3$ on Poincare sphere i.e. $S=2\alpha$.</td><td>0.8</td></tr>
</table>