Theoretical 1: Solution
Conductors in Conducting Liquid

1. Using Gauss law
$$\oint \mathbf{E}\cdot \mathbf{dA} = \frac{q}{\epsilon_0}. \tag{1}$$
From symetry we know that the electric field only has radial component. Choose a cylinder (with a line charge as the axis) as the Gaussian surface, we obtain
$$E.2\pi r l = \frac{\lambda l}{\epsilon_0}.$$
Simplify to obtain
$$\mathbf{E} = \hat{r}\frac{\lambda}{2\pi\epsilon_0 r}. \tag{2}$$

2. The potential is given by
$$\begin{aligned} V &= -\int_{\text{ref}}^{r} \mathbf{E}\cdot\mathbf{dl} \\ &= -\int_{\text{ref}}^{r} E.dr \\ V &= -\frac{\lambda}{2\pi\epsilon_0}\ln r + K, \end{aligned} \tag{3}$$
so $f(r) = -\frac{\lambda}{2\pi\epsilon_0}\ln r$. where K is a constant.

3. The potential from both line charges is a superposition of both potential

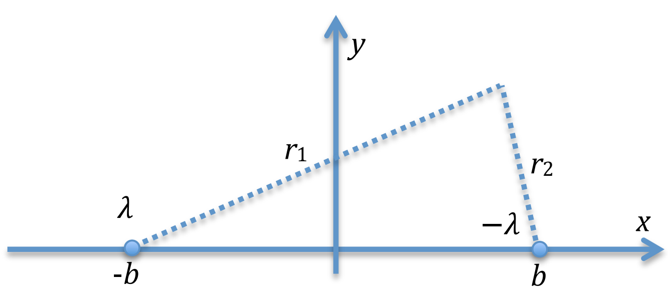


Figure 1: System with two line charges

$$V = -\frac{\lambda}{2\pi\epsilon_0}\ln r_1 + \frac{\lambda}{2\pi\epsilon_0}\ln r_2 \tag{4}$$
$$= \frac{\lambda}{2\pi\epsilon_0}\ln\frac{\sqrt{(b-x)^2+y^2}}{\sqrt{(b+x)^2+y^2}}$$
$$V = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{(b-x)^2+y^2}{(b+x)^2+y^2}. \tag{5}$$

We can rearrange eq.(5) in to:

$$\left(x-\left(\frac{1+\beta}{1-\beta}\right)\right)^2+y^2=b^2\left(\left(\frac{1+\beta}{1-\beta}\right)^2-1\right) \tag{6}$$

where $\beta=\exp\left(\frac{4\pi\epsilon_0 V}{\lambda}\right)$. For an arbitrary potential V, Eq. (6) is an equation of circle.

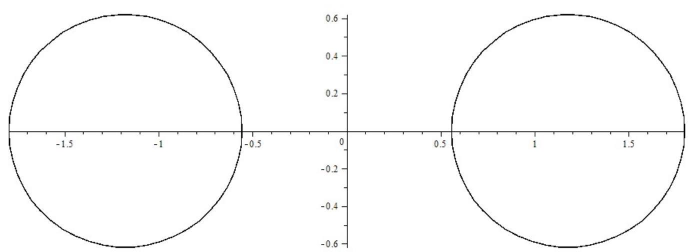


Figure 2: The equipotential surfaces with $b=1$, for $\beta=12.35$ (left) and $\beta=\frac{1}{12.35}$ (right)

4. From eq.(5) and eq.(6), we see that for any arbitrary potential V, the equipotential surfaces of these two equal but opposite lines charge, are cylindrical surfaces. From this observation, we can choose the specific position for each line charge in both cylinders so that the surface of each cylinder is an equipotential surface.

   Consider the following figure

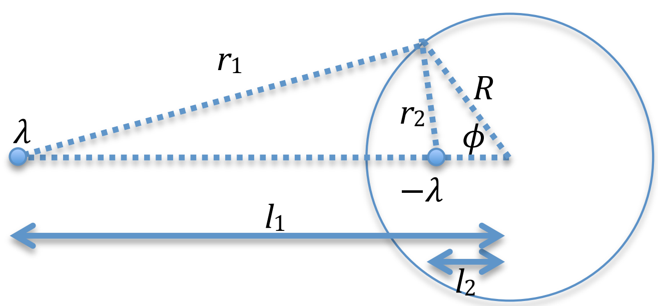


Figure 3: Two line charges with its equipotential surfaces

We would like to find a cylindrical equipotential surface enclose one line charge, let say the $-\lambda$ (if we could find the surface, by symmetry, we surely can find the identical one that enclose the line λ). The potential is given by

$$\begin{aligned} V &= -\frac{\lambda}{2\pi\epsilon_0}\ln r_1+\frac{\lambda}{2\pi\epsilon_0}\ln r_2 \\ &= -\frac{\lambda}{4\pi\epsilon_0}\ln(l_1^2+R^2-2l_1R\cos\phi)+\frac{\lambda}{4\pi\epsilon_0}\ln(l_2^2+R^2-2l_2R\cos\phi). \end{aligned} \tag{7}$$

Since the surface of the cylinder has to be the equipotential surface, so the potential should not depend on ϕ, i.e. $\frac{\partial V}{\partial \phi} = 0$.

$$-\frac{\lambda}{4\pi\epsilon_0}\frac{2l_1 R\sin\phi}{l_1^2+R^2-2l_1R\cos\phi}+\frac{\lambda}{4\pi\epsilon_0}\frac{2l_2R\sin\phi}{l_2^2+R^2-2l_2R\cos\phi}=0 \tag{8}$$

$$\frac{l_1}{l_1^2+R^2-2l_1R\cos\phi}=\frac{l_2}{l_2^2+R^2-2l_2R\cos\phi}$$

$$l_1^2l_2+R^2l_2-2l_1l_2R\cos\phi=l_1l_2^2+R^2l_1-2l_1l_2R\cos\phi$$

$$l_1l_2(l_1-l_2)=R^2(l_1-l_2)$$

$$l_1l_2=R^2. \tag{9}$$

From the data in the problem, we have

$$l_1+l_2=10a, \tag{10}$$

$$l_1l_2=9a^2. \tag{11}$$

Solve this quadratic equation to get

$$l_1=5a\pm4a. \tag{12}$$

However, since $l_1 > l_2$, we have

$$l_1=9a, \tag{13}$$

$$l_2=a. \tag{14}$$

Using this results on eq.(5), we have

$$V=\frac{\lambda}{4\pi\epsilon_0}\ln\frac{(4a-x)^2+y^2}{(4a+x)^2+y^2}. \tag{15}$$

This is the potential in all region except inside both cylinders. For cylinders at $x=-5a$, the potential is constant and equal to

$$V(x=-2a,y=0)=\frac{\lambda}{4\pi\epsilon_0}\ln\frac{(4a+2a)^2+0^2}{(4a-2a)^2+0^2}=\frac{\lambda}{2\pi\epsilon_0}\ln 3. \tag{16}$$

For cylinders at $x=5a$, the potential is constant and equal to

$$V(x=2a,y=0)=\frac{\lambda}{4\pi\epsilon_0}\ln\frac{(4a-2a)^2+0^2}{(4a+2a)^2+0^2}=-\frac{\lambda}{2\pi\epsilon_0}\ln 3. \tag{17}$$

The potential difference between both cylinders are

$$\Delta V=\frac{\lambda}{\pi\epsilon_0}\ln 3\equiv V_0. \tag{18}$$

Substituting this results in the potential equation, the potential outside the two cylinders are:

$$V = \frac{V_0}{4\ln 3}\ln\frac{(4a-x)^2+y^2}{(4a+x)^2+y^2}. \tag{19}$$

And the potential inside the cylinders are:

The potential inside the cylinder centered at $(x = 5a, y = 0)$ is $V = -V_0/2$.

The potential inside the cylinder centered at $(x = -5a, y = 0)$ is $V = V_0/2$.

5. From eq.(18), we have

$$V_0 = \frac{q}{l\pi\epsilon_0}\ln 3, \tag{20}$$

so we get

$$C = \frac{q}{V_0} = \frac{l\pi\epsilon_0}{\ln 3} \tag{21}$$

6. The electric field produces by both cylinders are

$$E_x = \frac{V_0}{2\ln 3}\left(\frac{4a+x}{(4a+x)^2+y^2} + \frac{4a-x}{(4a-x)^2+y^2}\right). \tag{22}$$

$$E_y = \frac{V_0}{2\ln 3}\left(\frac{y}{(4a+x)^2+y^2} - \frac{y}{(4a-x)^2+y^2}\right). \tag{23}$$

The volume current density is given by

$$\mathbf{J} = \sigma\mathbf{E} \tag{24}$$

To calculate the total current, we may choose to calculate the current that flow through the $x = 0$ plane. On this plane, there is no current in the y direction. The total current is given by

$$\begin{aligned} I &= \int \mathbf{J}\cdot\mathbf{dA} \qquad (25)\\ &= \int \sigma E_x l dy \\ &= \sigma l\frac{8aV_0}{2\ln 3}\int_{\infty}^{\infty}\frac{dy}{(4a)^2+y^2} \\ I &= \frac{V_0\pi\sigma l}{\ln 3} \qquad (26)\end{aligned}$$

7. The resistance is given by

$$R = \frac{V_0}{I} = \frac{\ln 3}{\pi\sigma l} \tag{27}$$

and therefore

$$RC = \frac{\epsilon_0}{\sigma} \tag{28}$$

8. Since the system has a high symmetry, we may use Ampere's law. The magnetic field should not have any z dependence, since the current has no z dependence.

   Figure 4 shows the current density $\mathbf{J}$ flow from one cylinder to the other cylinder. Choose an Ampere loop on a constant x plane in a symmetrical way, so that the first path is pointing in the positive z direction with constant y coordinate, the second path is pointing to the negative y direction with constant z coordinate. The third path is pointing to the negative z direction, but with constant $-y$ coordinate. The fourth path is pointing in the positive y direction with constant $-z$ coordinate.

   Having this path, we need to calculate the current that flow through the loop

$$\begin{aligned} I &= \int \mathbf{J} \cdot \mathbf{dA} \\ &= \int J_x l dy \\ &= \frac{V_0 \sigma l}{2 \ln 3} \int_{-y}^{y} \left(\frac{4a+x}{(4a+x)^2+y^2} + \frac{4a-x}{(4a-x)^2+y^2} \right) dy \end{aligned}$$

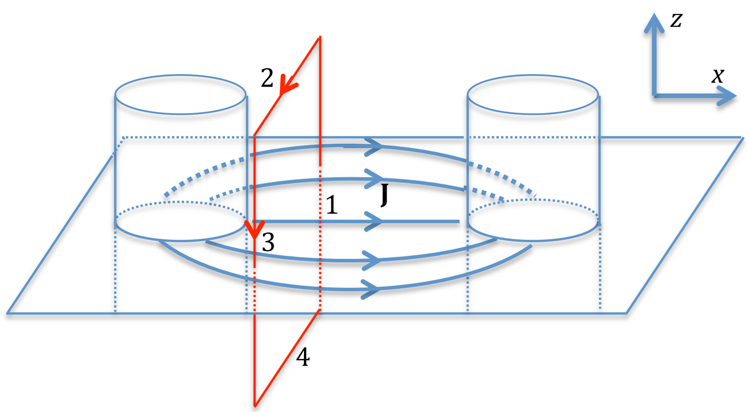


Figure 4: The Ampere loop

$$I = \frac{V_0 \sigma l}{\ln 3} \left(\arctan \frac{y}{4a+x} + \arctan \frac{y}{4a-x} \right) \tag{29}$$

Using the Ampere's law

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 I \tag{30}$$

$$2B_z l = \frac{\mu_0 V_0 \sigma l}{\ln 3} \left(\arctan \frac{y}{4a+x} + \arctan \frac{y}{4a-x} \right)$$

$$B_z = \mu_0 \frac{V_0 \sigma}{2 \ln 3} \left(\arctan \frac{y}{4a+x} + \arctan \frac{y}{4a-x} \right) \tag{31}$$

therefore

$$\mathbf{B} = \hat{z}\frac{\mu_0 V_0 \sigma}{2\ln 3}\left(\arctan\frac{y}{4a+x} + \arctan\frac{y}{4a-x}\right) \tag{32}$$

Theoretical 2: Solution
Relativistic Correction on GPS Satelitte

Part A. Single accelerated particle

1. The equation of motion is given by

$$F = \frac{d}{dt}(\gamma m v) \tag{1}$$
$$= \frac{mc\dot{\beta}}{(1-\beta^2)^{\frac{3}{2}}}$$
$$F = \gamma^3 m a, \tag{2}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and $\beta = \frac{v}{c}$. So the acceleration is given by

$$a = \frac{F}{\gamma^3 m}. \tag{3}$$

2. Eq.(3) can be rewritten as

$$c\frac{d\beta}{dt} = \frac{F}{\gamma^3 m}$$
$$\int_0^\beta \frac{d\beta}{(1-\beta^2)^{\frac{3}{2}}} = \frac{F}{mc}\int_0^t dt$$
$$\frac{\beta}{\sqrt{1-\beta^2}} = \frac{Ft}{mc} \tag{4}$$
$$\beta = \frac{\frac{Ft}{mc}}{\sqrt{1+\left(\frac{Ft}{mc}\right)^2}}. \tag{5}$$

3. Using Eq.(5), we get

$$\int_0^x dx = \int_0^t \frac{Ftdt}{m\sqrt{1+\left(\frac{Ft}{mc}\right)^2}}$$
$$x = \frac{mc^2}{F}\left(\sqrt{1+\left(\frac{Ft}{mc}\right)^2} - 1\right). \tag{6}$$

4. Consider the following systems, a frame S' is moving with respect to another frame S, with velocity u in the x direction. If a particle is moving in the S' frame with velocity v' also in x direction, then the particle velocity in the S frame is given by

$$v = \frac{u+v'}{1+\frac{uv'}{c^2}}. \tag{7}$$

If the particles velocity changes with respect to the S' frame, then the velocity in the S frame is also change according to

$$dv = \frac{dv'}{1+\frac{uv'}{c^2}} - \frac{u+v'}{\left(1+\frac{uv'}{c^2}\right)^2}\frac{udv'}{c^2}$$

$$dv = \frac{1}{\gamma^2}\frac{dv'}{\left(1+\frac{uv'}{c^2}\right)^2}. \tag{8}$$

The time in the S' frame is t', so the time in the S frame is given by

$$t = \gamma\left(t' + \frac{ux'}{c^2}\right), \tag{9}$$

so the time change in the S' frame will give a time change in the S frame as follow

$$dt = \gamma dt'\left(1 + \frac{uv'}{c^2}\right). \tag{10}$$

The acceleration in the S frame is given by

$$a = \frac{dv}{dt} = \frac{a'}{\gamma^3}\frac{1}{\left(1+\frac{uv'}{c^2}\right)^3}. \tag{11}$$

If the S' frame is the proper frame, then by definition the velocity $v' = 0$. Substitute this to the last equation, we get

$$a = \frac{a'}{\gamma^3}. \tag{12}$$

Combining Eq.(3) and Eq.(12), we get

$$a' = \frac{F}{m} \equiv g. \tag{13}$$

5. Eq.(3) can also be rewritten as

$$c\frac{d\beta}{\gamma d\tau} = \frac{g}{\gamma^3} \tag{14}$$

$$\int_0^\beta \frac{d\beta}{1-\beta^2} = \frac{g}{c}\int_0^\tau d\tau$$

$$\ln\left(\frac{1}{\sqrt{1-\beta^2}} + \frac{\beta}{\sqrt{1-\beta^2}}\right) = \frac{g\tau}{c} \tag{15}$$

$$\sqrt{\frac{1+\beta}{1-\beta}} = e^{\frac{g\tau}{c}}$$

$$\beta\left(e^{\frac{g\tau}{c}} + e^{-\frac{g\tau}{c}}\right) = e^{\frac{g\tau}{c}} - e^{-\frac{g\tau}{c}}$$

$$\beta = \tanh\frac{g\tau}{c}. \tag{16}$$

6. The time dilation relation is

$$dt = \gamma d\tau. \tag{17}$$

From eq.(16), we have

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\frac{g\tau}{c}. \tag{18}$$

Combining this equations, we get

$$\int_0^t dt = \int_0^\tau d\tau \cosh\frac{g\tau}{c}$$
$$t = \frac{c}{g}\sinh\frac{g\tau}{c}. \tag{19}$$

Part B. Flight Time

1. When the clock in the origin time is equal to t_0, it emits a signal that contain the information of its time. This signal will arrive at the particle at time t, while the particle position is at $x(t)$. We have

$$c(t-t_0) = x(t) \tag{20}$$
$$t - t_0 = \frac{c}{g}\left(\sqrt{1+\left(\frac{gt}{c}\right)^2} - 1\right)$$
$$t = \frac{t_0}{2}\frac{2-\frac{gt_0}{c}}{1-\frac{gt_0}{c}}. \tag{21}$$

When the information arrive at the particle, the particle's clock has a reading according to eq.(19). So we get

$$\frac{c}{g}\sinh\frac{g\tau}{c} = \frac{t_0}{2}\frac{2-\frac{gt_0}{c}}{1-\frac{gt_0}{c}}$$
$$0 = \frac{1}{2}\left(\frac{gt_0}{c}\right)^2 - \frac{gt_0}{c}\left(1+\sinh\frac{g\tau}{c}\right) + \sinh\frac{g\tau}{c}$$
$$\frac{gt_0}{c} = 1 + \sinh\frac{g\tau}{c} \pm \cosh\frac{g\tau}{c}. \tag{22}$$

Using initial condition $t = 0$ when $\tau = 0$, we choose the negative sign

$$\frac{gt_0}{c} = 1 + \sinh\frac{g\tau}{c} - \cosh\frac{g\tau}{c}$$
$$t_0 = \frac{c}{g}\left(1 - e^{-\frac{g\tau}{c}}\right). \tag{23}$$

As $\tau \to \infty$, $t_0 = \frac{c}{g}$. So the clock reading will freeze at this value.

2. When the particles clock has a reading τ_0, its position is given by eq.(6), and the time t_0 is given by eq.(19). Combining this two equation, we get

$$x = \frac{c^2}{g}\left(\sqrt{1+\sinh^2\frac{g\tau_0}{c}} - 1\right). \tag{24}$$

The particle's clock reading is then sent to the observer at the origin. The total time needed for the information to arrive is given by

$$\begin{aligned} t &= \frac{c}{g}\sinh\frac{g\tau_0}{c} + \frac{x}{c} \qquad (25)\\ &= \frac{c}{g}\left(\sinh\frac{g\tau_0}{c} + \cosh\frac{g\tau_0}{c} - 1\right) \\ t &= \frac{c}{g}\left(e^{\frac{g\tau_0}{c}} - 1\right) \qquad (26)\\ \tau_0 &= \frac{c}{g}\ln\left(\frac{gt}{c} + 1\right). \qquad (27) \end{aligned}$$

The time will not freeze.

Part C. Minkowski Diagram

1. The figure below show the setting of the problem.

The line AB represents the stick with proper length equal L in the S frame.
The length AB is equal to $\sqrt{\frac{1-\beta^2}{1+\beta^2}}L$ in the S' frame.
The stick length in the S' frame is represented by the line AC

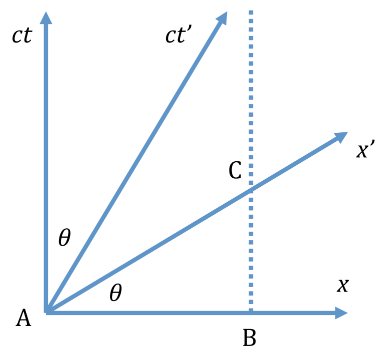


Figure 1: Minkowski Diagram

$$\text{AC} = \frac{\text{AB}}{\cos\theta} = \sqrt{1-\beta^2}L. \tag{28}$$

2. The position of the particle is given by eq.(6).

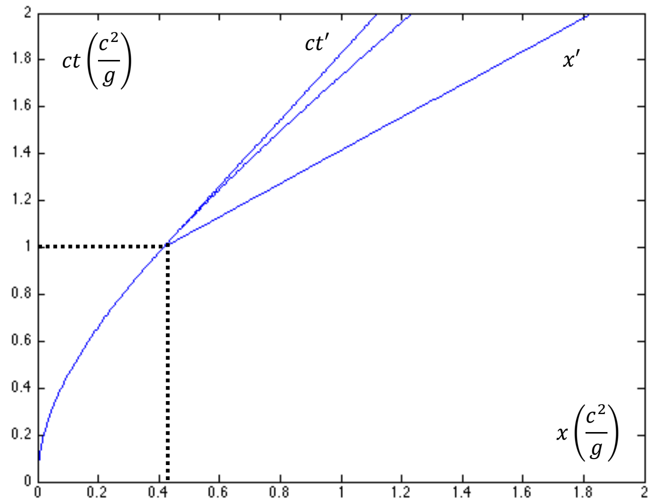


Figure 2: Minkowski Diagram

Part D. Two Accelerated Particles

1. $\tau_B = \tau_A$.

2. From the diagram, we have

$$\tan\theta = \beta = \frac{ct_2 - ct_1}{x_2 - x_1}. \tag{29}$$

Using eq.(6), and eq.(19) along with the initial condition, we get

$$x_1 = \frac{c^2}{g}\left(\cosh\frac{g\tau_1}{c} - 1\right), \tag{30}$$

$$x_2 = \frac{c^2}{g}\left(\cosh\frac{g\tau_2}{c} - 1\right) + L. \tag{31}$$

Using eq.(16), eq.(19), eq.(30) and eq.(31), we obtain

$$\begin{aligned}
\tanh\frac{g\tau_1}{c} &= \frac{c\left(\frac{c}{g}\sinh\frac{g\tau_2}{c} - \frac{c}{g}\sinh\frac{g\tau_1}{c}\right)}{L + \frac{c^2}{g}\left(\cosh\frac{g\tau_2}{c} - 1\right) - \frac{c^2}{g}\left(\cosh\frac{g\tau_1}{c} - 1\right)} \\
&= \frac{\sinh\frac{g\tau_2}{c} - \sinh\frac{g\tau_1}{c}}{\frac{gL}{c^2} + \cosh\frac{g\tau_2}{c} - \cosh\frac{g\tau_1}{c}} \\
\frac{gL}{c^2}\sinh\frac{g\tau_1}{c} &= \sinh\frac{g\tau_2}{c}\cosh\frac{g\tau_1}{c} - \cosh\frac{g\tau_2}{c}\sinh\frac{g\tau_1}{c} \\
\frac{gL}{c^2}\sinh\frac{g\tau_1}{c} &= \sinh\frac{g}{c}\left(\tau_2 - \tau_1\right).
\end{aligned} \tag{32}$$

So $C_1 = \frac{gL}{c^2}$.

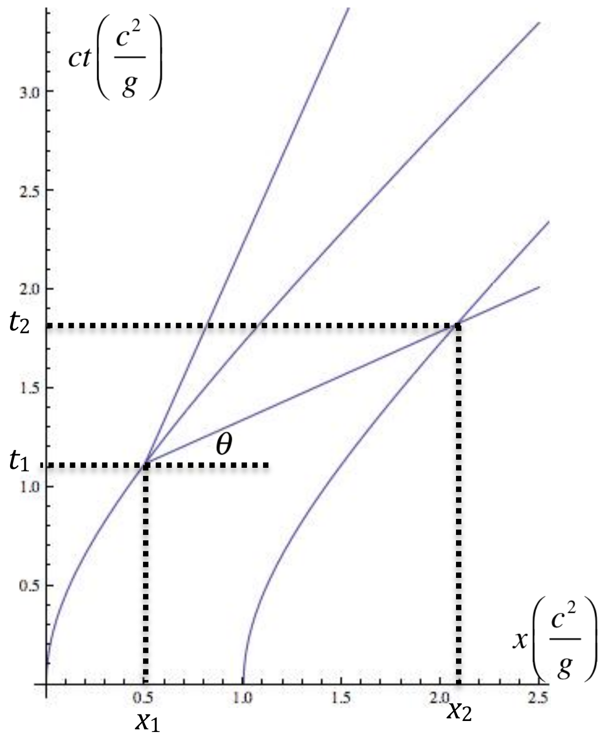


Figure 3: Minkowski Diagram for two particles

3. From the length contraction, we have

$$L' = \frac{x_2 - x_1}{\gamma_1} \tag{33}$$

$$\frac{dL'}{d\tau_1} = \left(\frac{dx_2}{d\tau_2}\frac{d\tau_2}{d\tau_1} - \frac{dx_1}{d\tau_1}\right)\frac{1}{\gamma_1} - \frac{x_2 - x_1}{\gamma_1^2}\frac{d\gamma_1}{d\tau_1}. \tag{34}$$

Take derivative of eq.(30), eq.(31) and eq.(32), we get

$$\frac{dx_1}{d\tau_1} = c\sinh\frac{g\tau_1}{c}, \tag{35}$$

$$\frac{dx_2}{d\tau_2} = c\sinh\frac{g\tau_2}{c}, \tag{36}$$

$$\frac{gL}{c^2}\cosh\frac{g\tau_1}{c} = \cosh\frac{g}{c}\left(\tau_2 - \tau_1\right)\left(\frac{d\tau_2}{d\tau_1} - 1\right). \tag{37}$$

The last equation can be rearrange to get

$$\frac{d\tau_2}{d\tau_1} = \frac{\frac{gL}{c^2}\cosh\frac{g\tau_1}{c}}{\cosh\frac{g}{c}\left(\tau_2 - \tau_1\right)} + 1. \tag{38}$$

From eq.(29), we have

$$x_2 - x_1 = \frac{c(t_2 - t_1)}{\beta_1} = \frac{c}{\tanh\frac{g\tau_1}{c}}\left(\frac{c}{g}\sinh\frac{g\tau_2}{c} - \frac{c}{g}\sinh\frac{g\tau_1}{c}\right). \tag{39}$$

Combining all these equations, we get

$$\begin{aligned}
\frac{dL_1}{d\tau_1} &= \left(c\sinh\frac{g\tau_2}{c}\frac{\frac{gL}{c^2}\cosh\frac{g\tau_1}{c}}{\cosh\frac{g}{c}(\tau_2-\tau_1)} + c\sinh\frac{g\tau_2}{c} - c\sinh\frac{g\tau_1}{c}\right)\frac{1}{\cosh\frac{g\tau_1}{c}} \\
&\quad - \frac{c^2}{g}\left(\sinh\frac{g\tau_2}{c} - \sinh\frac{g\tau_1}{c}\right)\frac{1}{\tanh\frac{g\tau_1}{c}}\frac{1}{\cosh^2\frac{g\tau_1}{c}}\frac{g}{c}\sinh\frac{g\tau_1}{c} \\
\frac{dL_1}{d\tau_1} &= \frac{gL}{c}\frac{\sinh\frac{g\tau_2}{c}}{\cosh\frac{g}{c}(\tau_2-\tau_1)}. \qquad (40)
\end{aligned}$$

So $C_2 = \frac{gL}{c}$.

Part E. Uniformly Accelerated Frame

1. Distance from a certain point x_p according to the particle's frame is

$$L' = \frac{x - x_p}{\gamma} \tag{41}$$

$$L' = \frac{\frac{c^2}{g_1}\left(\cosh\frac{g_1\tau}{c} - 1\right) - x_p}{\cosh\frac{g_1\tau}{c}}$$

$$L' = \frac{c^2}{g_1} - \frac{\frac{c^2}{g_1} + x_p}{\cosh\frac{g_1\tau}{c}}. \tag{42}$$

For L' equal constant, we need $x_p = -\frac{c^2}{g_1}$.

2. **First method:** If the distance in the S' frame is constant $= L$, then in the S frame the length is

$$L_s = L\sqrt{\frac{1+\beta^2}{1-\beta^2}}. \tag{43}$$

So the position of the second particle is

$$\begin{aligned}
x_2 &= x_1 + L_s\cos\theta \qquad (44) \\
&= \frac{c^2}{g_1}\left(\sqrt{1+\left(\frac{g_1t_1}{c}\right)^2} - 1\right) + L\sqrt{1+\left(\frac{g_1t_1}{c}\right)} \\
x_2 &= \left(\frac{c^2}{g_1} + L\right)\sqrt{1+\left(\frac{g_1t_1}{c}\right)^2} - \frac{c^2}{g_1}. \qquad (45)
\end{aligned}$$

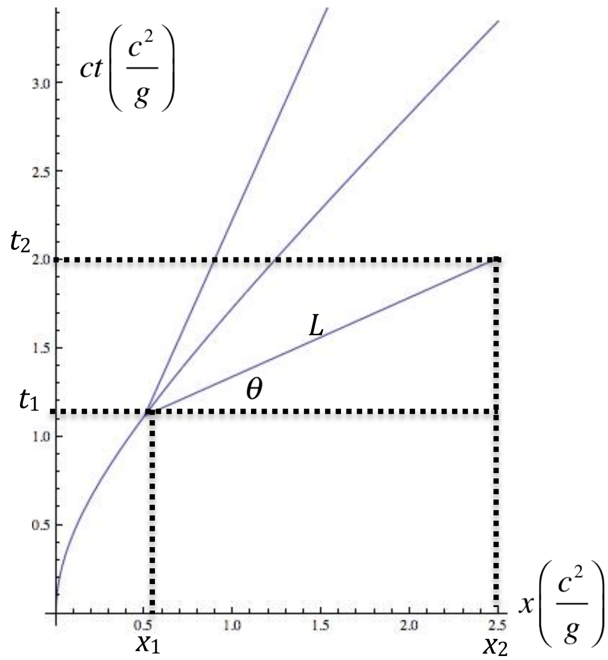


Figure 4: Minkowski Diagram for two particles

The time of the second particle is

$$\begin{aligned} ct_2 &= ct_1 + L_s \sin\theta \\ &= ct_1 + L\sqrt{\frac{1+\beta^2}{1-\beta^2}}\frac{\beta}{\sqrt{1+\beta^2}} \end{aligned} \tag{46}$$

$$ct_2 = t_1\left(c + \frac{g_1 L}{c}\right). \tag{47}$$

Substitute eq.(47) to eq.(45) to get

$$x_2 = \left(\frac{c^2}{g_1} + L\right)\sqrt{1 + \left(\frac{g_1}{c}\frac{t_2}{1+\frac{g_1 L}{c^2}}\right)^2} - \frac{c^2}{g_1}$$

$$x_2 = \left(\frac{c^2}{g_1} + L\right)\sqrt{1 + \left(\frac{g_1}{1+\frac{g_1 L}{c^2}}\frac{t_2}{c}\right)^2} - \frac{c^2}{g_1}. \tag{48}$$

From the last equation, we can identify

$$g_2 \equiv \frac{g_1}{1+\frac{g_1 L}{c^2}}. \tag{49}$$

As for confirmation, we can subsitute this relation to the second particle position to get

$$x_2 = \frac{c^2}{g_2}\sqrt{1+\left(\frac{g_2 t_2}{c}\right)^2} - \frac{c^2}{g_1}. \tag{50}$$

Second method: In this method, we will choose g_2 such that the special point like the one descirbe in the question 1 is exactly the same as the similar point for the proper acceleration g_1.

For first particle, we have $x_{p1}g_1 = c^2$

For second particle, we have $(L + x_{p1})g_2 = c^2$

Combining this two equations, we get

$$g_2 = \frac{c^2}{L+\frac{c^2}{g_1}}$$

$$g_2 = \frac{g_1}{1+\frac{g_1 L}{c^2}}. \tag{51}$$

3. The relation between the time in the two particles is given by eq.(47)

$$t_2 = t_1\left(1+\frac{g_1 L}{c^2}\right)$$

$$\frac{c^2}{g_2}\sinh\frac{g_2\tau_2}{c} = \frac{c^2}{g_1}\sinh\frac{g_1\tau_1}{c}\left(1+\frac{g_1 L}{c^2}\right)$$

$$\sinh\frac{g_2\tau_2}{c} = \sinh\frac{g_1\tau_1}{c}$$

$$g_2\tau_2 = g_1\tau_1 \tag{52}$$

$$\frac{d\tau_2}{d\tau_1} = \frac{g_1}{g_2} = 1+\frac{g_1 L}{c^2}. \tag{53}$$

Part F. Correction for GPS

1. From Newtons Law

$$\frac{GMm}{r^2} = m\omega^2 r \tag{54}$$

$$r = \left(\frac{gR^2T^2}{4\pi^2}\right)^{\frac{1}{3}} \tag{55}$$

$$r = 2.66\times10^7\text{m}.$$

The velocity is given by

$$v = \omega r = \left(\frac{2\pi g R^2}{T}\right)^{\frac{1}{3}} \tag{56}$$

$$= 3.87\times10^3\text{m/s}.$$

2. The general relativity effect is

$$\frac{d\tau_g}{dt} = 1 + \frac{\Delta U}{mc^2} \tag{57}$$

$$\frac{d\tau_g}{dt} = 1 + \frac{gR^2}{c^2}\frac{R-r}{Rr}. \tag{58}$$

After one day, the difference is

$$\begin{aligned}\Delta\tau_g &= \frac{gR^2}{c^2}\frac{R-r}{Rr}\Delta T \\ &= 4.55\times 10^{-5}\text{s.}\end{aligned} \tag{59}$$

The special relativity effect is

$$\begin{aligned}\frac{d\tau_s}{dt} &= \sqrt{1-\frac{v^2}{c^2}} \\ &= \sqrt{1-\left(\left(\frac{2\pi gR^2}{T}\right)^{\frac{2}{3}}\right)\frac{1}{c^2}}\end{aligned} \tag{60}$$

$$\approx 1-\frac{1}{2}\left(\left(\frac{2\pi gR^2}{T}\right)^{\frac{2}{3}}\right)\frac{1}{c^2}. \tag{61}$$

After one day, the difference is

$$\begin{aligned}\Delta\tau_s &= -\frac{1}{2}\left(\left(\frac{2\pi gR^2}{T}\right)^{\frac{2}{3}}\right)\frac{1}{c^2}\Delta T \\ &= -7.18\times 10^{-6}\text{s.}\end{aligned} \tag{62}$$

The satelite's clock is faster with total $\Delta\tau = \Delta\tau_g + \Delta\tau_s = 3.83\times 10^{-5}$s.

3. $\Delta L = c\Delta\tau = 1.15\times 10^4\text{m} = 11.5\text{km}$.

Theoretical 3: Solution

Physics of Spin

Part A. Larmor Precession

1. From the two equations given in the text, we obtain the relation

$$\frac{d\boldsymbol{\mu}}{dt} = -\gamma\boldsymbol{\mu} \times \mathbf{B}. \tag{1}$$

Taking the dot product of eq (1). with $\boldsymbol{\mu}$, we can prove that

$$\begin{aligned} \boldsymbol{\mu} \cdot \frac{d\boldsymbol{\mu}}{dt} &= -\gamma\boldsymbol{\mu} \cdot (\boldsymbol{\mu} \times \mathbf{B}), \\ \frac{d|\boldsymbol{\mu}|^2}{dt} &= 0, \\ \mu = |\boldsymbol{\mu}| &= \text{const.} \end{aligned} \tag{2}$$

Taking the dot product of eq. (1) with $\mathbf{B}$, we also prove that

$$\begin{aligned} \mathbf{B} \cdot \frac{d\boldsymbol{\mu}}{dt} &= -\gamma\mathbf{B} \cdot (\boldsymbol{\mu} \times \mathbf{B}), \\ \mathbf{B} \cdot \frac{d\boldsymbol{\mu}}{dt} &= 0, \\ \mathbf{B} \cdot \boldsymbol{\mu} &= \text{const.} \end{aligned} \tag{3}$$

An acute reader will notice that our master equation in (1) is identical to the equation of motion for a charged particle in a magnetic field

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{v} \times \mathbf{B}. \tag{4}$$

Hence, the same argument for a charged particle in magnetic field can be applied in this case.

2. For a magnetic moment making an angle of ϕ with $\mathbf{B}$,

$$\begin{aligned} \frac{d\boldsymbol{\mu}}{dt} &= -\gamma\boldsymbol{\mu} \times \mathbf{B}, \\ |\boldsymbol{\mu}| \sin\phi \frac{d\theta}{dt} &= \gamma |\boldsymbol{\mu}| B_0 \sin\phi, \\ \omega_0 = \frac{d\theta}{dt} &= \gamma B_0. \end{aligned} \tag{5}$$

Part B. Rotating frame

1. Using the relation given in the text, it is easily shown that

$$\begin{aligned} \left(\frac{d\boldsymbol{\mu}}{dt}\right)_{\text{rot}} &= \left(\frac{d\boldsymbol{\mu}}{dt}\right)_{\text{lab}} - \boldsymbol{\omega} \times \boldsymbol{\mu} \\ &= -\gamma\boldsymbol{\mu} \times \mathbf{B} - \omega\mathbf{k}' \times \boldsymbol{\mu} \\ &= -\gamma\boldsymbol{\mu} \times \left(\mathbf{B} - \frac{\omega}{\gamma}\mathbf{k}'\right) \\ &= -\gamma\boldsymbol{\mu} \times \mathbf{B}_{\text{eff}}. \end{aligned} \tag{6}$$

Note that $\mathbf{k}$ is equal to $\mathbf{k}'$ as observed in the rotating frame.

2. The new precession frequency as viewed on the rotating frame S' is

$$\begin{aligned}\vec{\Delta} &= (\omega_0 - \omega)\,\mathbf{k}', \\ \Delta &= \gamma B_0 - \omega. \end{aligned} \tag{7}$$

3. Since the magnetic field as viewed in the rotating frame is $\mathbf{B} = B_0\mathbf{k}' + b\mathbf{i}'$,

$$\mathbf{B}_{\text{eff}} = \mathbf{B} - \omega/\gamma\mathbf{k}' = \left(B_0 - \frac{\omega}{\gamma}\right)\mathbf{k}' + b\mathbf{i}',$$

and

$$\begin{aligned}\Omega &= \gamma\left|\mathbf{B}_{\text{eff}}\right|, \\ &= \gamma\sqrt{\left(B_0 - \frac{\omega}{\gamma}\right)^2 + b^2}. \end{aligned} \tag{8}$$

4. In this case, the effective magnetic field becomes

$$\begin{aligned}\mathbf{B}_{\text{eff}} &= \mathbf{B} - \omega/\gamma\mathbf{k}' \\ &= \left(B_0 - \frac{\omega}{\gamma}\right)\mathbf{k}' + b(\cos 2\omega t\mathbf{i}' - \sin 2\omega t\mathbf{j}') \end{aligned} \tag{9}$$

which has a time average of $\overline{\mathbf{B}_{\text{eff}}} = \left(B_0 - \frac{\omega}{\gamma}\right)\mathbf{k}'$.

Part C. Rabi oscillation

1. The oscillating field can be considered as a superposition of two oppositely rotating field:

$$2b\cos\omega_0 t\mathbf{i} = b\left(\cos\omega_0 t\mathbf{i} + \sin\omega_0 t\mathbf{j}\right) + b\left(\cos\omega_0 t\mathbf{i} - \sin\omega_0 t\mathbf{j}\right),$$

which gives an effective field of (with $\omega = \omega_0 = \gamma B_0$):

$$\mathbf{B}_{\text{eff}} = \left(B_0 - \frac{\omega}{\gamma}\right)\mathbf{k}' + b\mathbf{i}' + b(\cos 2\omega_0 t\mathbf{i}' - \sin 2\omega_0 t\mathbf{j}').$$

Since $\omega_0 \gg \gamma b$, the rotation of the term $b(\cos 2\omega_0 t\mathbf{i}' - \sin 2\omega_0 t\mathbf{j}')$ is so fast compared to the frequency γb. This means that we can take the approximation

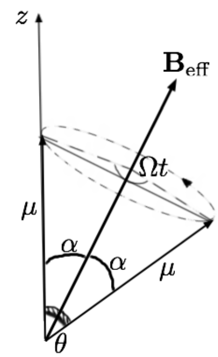
$$\mathbf{B}_{\text{eff}} \approx \left(B_0 - \frac{\omega}{\gamma}\right)\mathbf{k}' + b\mathbf{i}' = b\mathbf{i}', \tag{10}$$

where the magnetic moment precesses with frequency $\Omega = \gamma b$.

As $\Omega = \gamma b \ll \omega_0$, the magnetic moment does not "feel" the rotating term $b\left(\cos 2\omega_0 t\mathbf{i}' - \sin 2\omega_0 t\mathbf{j}'\right)$ which averaged to zero.

2. Since the angle α that $\boldsymbol{\mu}$ makes with $\mathbf{B}_{\text{eff}}$ stays constant and $\boldsymbol{\mu}$ is initially oriented along the z axis, α is also the angle between $\mathbf{B}_{\text{eff}}$ and the z axis which is

$$\tan\alpha = \frac{b}{B_0 - \frac{\omega}{\gamma}}. \tag{11}$$



From the geometry of the system, we can show that ($\cos\theta = \mu_z/\mu$):

$$\begin{aligned}
2\mu\sin\frac{\theta}{2} &= 2\mu\sin\alpha\sin\frac{\Omega t}{2}, \\
\sin^2\frac{\theta}{2} &= \sin^2\alpha\sin^2\frac{\Omega t}{2}, \\
\frac{1-\cos\theta}{2} &= \sin^2\alpha\frac{1-\cos\Omega t}{2}, \\
\cos\theta &= 1-\sin^2\alpha+\sin^2\alpha\cos\Omega t, \\
\cos\theta &= \cos^2\alpha+\sin^2\alpha\cos\Omega t.
\end{aligned}$$

So, the projected magnetic moment along the z axis is $\mu_z(t) = \mu\cos\theta$ and the magnetization is

$$M = N\mu_z = N\mu\left(\cos^2\alpha + \sin^2\alpha\cos\Omega t\right). \tag{12}$$

Note that the magnetization does not depend on the reference frame S or S' (μ_z has the same value viewed in both frames).

Taking $\omega = \omega_0 = \gamma B_0$, the angle α is 90^0 and $M = N\mu\cos\Omega t$.

3. From the relations

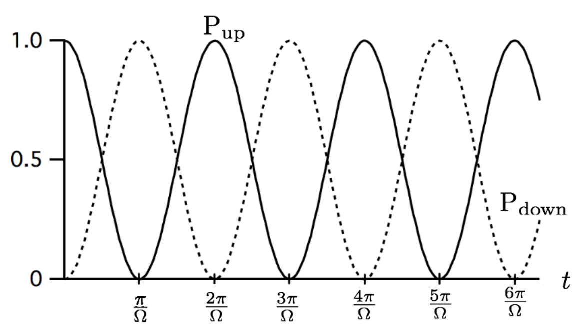
$$\begin{aligned}
P_\uparrow - P_\downarrow &= \frac{\mu_z}{\mu} = \cos\theta, \\
P_\uparrow + P_\downarrow &= 1,
\end{aligned}$$

we obtain the results ($\omega = \omega_0$)

$$
\begin{aligned}
P_{\downarrow} &= \frac{1-\cos\theta}{2} \\
&= \frac{1-\cos^2\alpha - \sin^2\alpha\cos\Omega t}{2} \\
&= \sin^2\alpha\frac{1-\cos\Omega t}{2} \\
&= \frac{b^2}{\left(B_0 - \frac{\omega}{\gamma}\right)^2 + b^2}\sin^2\frac{\Omega t}{2} \\
&= \sin^2\frac{\Omega t}{2},
\end{aligned}
\tag{13}
$$

and

$$
P_{\uparrow} = \frac{b^2}{\left(B_0 - \frac{\omega}{\gamma}\right)^2 + b^2}\cos^2\frac{\Omega t}{2} = \cos^2\frac{\Omega t}{2}. \tag{14}
$$



Part D. Measurement incompatibility

1. In the x direction, the uncertainty in position due to the screen opening is Δx. According to the uncertainty principle, the atom momentum uncertainty Δp_x is given by

$$\Delta p_x \approx \frac{\hbar}{\Delta x},$$

which translates into an uncertainty in the x velocity of the atom,

$$v_x \approx \frac{\hbar}{m\Delta x}.$$

Consequently, during the time of flight t of the atoms through the device, the uncertainty in the width of the beam will grow by an amount δx given by

$$\delta x = \Delta v_x t \approx \frac{\hbar}{m\Delta x}t.$$

So, the width of the beams is growing linearly in time. Meanwhile, the two beams are separating at a rate determined by the force F_x and the separation between the beams after a time t becomes

$$d_x = 2 \times \frac{1}{2}\frac{F_x}{m}t^2 = \frac{1}{m}\left|\mu_x\right| Ct^2.$$

In order to be able to distinguish which beam a particle belongs to, the separation of the two beams must be greater than the widths of the beams; otherwise the two beams will overlap and it will be impossible to know what the x component of the atom spin is. Thus, the condition must be satisfied is

$$\begin{aligned} d_x &\gg \delta x, \\ \frac{1}{m}\left|\mu_x\right| Ct^2 &\gg \frac{\hbar}{m\Delta x}t, \\ \frac{1}{\hbar}\left|\mu_x\right| \Delta x Ct &\gg 1. \end{aligned} \tag{15}$$

2. As the atoms pass through the screen, the variation of magnetic field strength across the beam width experienced by the atoms is

$$\Delta B = \Delta x \frac{dB}{dx} = C\Delta x.$$

This means the atoms will precess at rates covering a range of values $\Delta\omega$ given by

$$\Delta\omega = \gamma\Delta B = \frac{\mu_z}{\hbar}\Delta B = \frac{\left|\mu_x\right|}{\hbar}C\Delta x,$$

and, if previous condition in measuring μ_x is satisfied,

$$\Delta\omega t \gg 1. \tag{16}$$

In other words, the spread in the angle $\Delta\omega t$ through which the magnetic moments precess is so large that the z component of the spin is completely randomized or the measurement uncertainty is very large.