Question 1

The fractional quantum Hall effect (FQHE) was discovered by D. C. Tsui and H. Stormer at Bell Labs in 1981. In the experiment electrons were confined in two dimensions on the GaAs side by the interface potential of a GaAs/AlGaAs heterojunction fabricated by A. C. Gossard (here we neglect the thickness of the two-dimensional electron layer). A strong uniform magnetic field B was applied perpendicular to the two-dimensional electron system. As illustrated in Figure 1, when a current I was passing through the sample, the voltage V_H across the current path exhibited an unexpected quantized plateau (corresponding to a Hall resistance $R_H = 3h/e^2$) at sufficiently low temperatures. The appearance of the plateau would imply the presence of fractionally charged quasiparticles in the system, which we analyze below. For simplicity, we neglect the scattering of the electrons by random potential, as well as the electron spin.

(a) In a classical model, two-dimensional electrons behave like charged billiard balls on a table. In the GaAs/AlGaAs sample, however, the mass of the electrons is reduced to an effective m^* due to their interaction with ions.

(i) (**2 point**) Write down the equation of motion of an electron in perpendicular electric field $\vec{E} = -E_y\hat{y}$ and magnetic field $\vec{B} = B\hat{z}$.

Solution: An electron with charge $-e$ ($e > 0$) experiences the Lorentz force due to the perpendicular magnetic field and the electric force

$$m^*\frac{d\vec{v}}{dt} = -e\left(\vec{v} \times \vec{B} + \vec{E}\right)$$

where $\vec{v}$ is the velocity of the electron.

Grading: 1 point for writing down the electric force and the magnetic force correct, and 1 point for writing down the effective mass and the acceleration correct.

(ii) (**1 point**) Determine the velocity v_s of the electrons in the stationary case.

Solution: In the stationary regime, the acceleration vanishes. Hence

$$\vec{v}_s \times \vec{B} + \vec{E} = 0$$

The velocity can be expressed as

$$\vec{v}_s = \frac{\vec{E} \times \vec{B}}{B^2}$$

whose magnitude is simply $v_s = E/B$.

Grading: Either writing down the correct magnitude of the velocity or its vector form is sufficient for the 1 point.

(iii) (**1 point**) Which direction is the velocity pointing at?

Solution: The velocity $\vec{v}_s$ should be perpendicular to both the magnetic field and the electric field. If $\vec{B}$ is in the z direction and $\vec{E}$ in the $-y$ direction, as given by the problem, $\vec{v}_s$ is in the $-x$ direction, generating a charge current in the x direction.

Grading: 1 point for the correct direction.

(b) (**2 points**) The Hall resistance is defined as $R_H = V_H/I$. In the classical model, find R_H as a function of the number of the electrons N and the magnetic flux $\phi = BA = BWL$, where A is the area of the sample, and W

and L the effective width and length of the sample, respectively.

Solution: The Hall voltage $V_H = E_y W$. The current in the $-x$ direction is

$$I = \frac{\Delta Q}{\Delta t} = \frac{Ne}{L/v_s} = \frac{Ne}{L}\frac{E_y}{B} = e\frac{N}{\phi}V_H$$

Therefore,

$$R_H = \frac{V_H}{I} = \frac{1}{e}\frac{\phi}{N}$$

Grading: 1 point for the final expression and 1 point for writing down the expression for relating I with the number of electrons and their stationary velocity (hence the electric field and the magnetic field).

(c) (**2 points**) We know that electrons move in circular orbitss in the magnetic field. In the quantum mechanical picture, the impinging magnetic field B could be viewed as creating tiny whirlpools, so-called vortices, in the sea of electrons–one whirlpool for each flux quantum h/e of the magnetic field, where h is the Planck's constant and e the elementary charge of an electron. For the case of $R_H = 3h/e^2$, which was discovered by Tsui and Stormer, derive the ratio of the number of the electrons N to the number of the flux quanta N_ϕ, known as the filling factor ν.

Solution: The Hall resistance can be rewritten as

$$R_H = \frac{1}{e}\frac{\phi}{N} = \frac{h}{e^2}\frac{\phi/(h/e)}{N} = \frac{h}{e^2}\frac{N_\phi}{N}$$

At the plateau, $\nu = N/N_\phi = 1/3$.

Grading: 1 point for the final expression.

(d) (**2 points**) It turns out that binding an integer number of vortices ($n > 1$) with each electron generates a bigger surrounding whirlpool, hence pushes away all other electrons. Therefore, the system can considerably reduce its electrostatic Coulomb energy at the corresponding filling factor. Determine the scaling exponent α of the amount of energy gain for each electron $\Delta U(B) \propto B^{\alpha}$.

Solution: The average distance between electrons can be written as fl_0, where

$$l_0 = \sqrt{\frac{LW}{N}} = \sqrt{\frac{\phi}{NB}} = \sqrt{\frac{h}{\nu e B}}$$

and f is a dimensionless constant that is determined by the electron distribution (or, quantum mechanically, wave function). Binding multiple vortices with an electron effectively reduces the probability of other electrons getting close. Therefore, the electrons optimize their distribution in such a way that their average distance increases from $f_1 l_0$ to $f_2 l_0$ ($f_1 < f_2$). One expect the Coulomb energy gain per electron is proportional to

$$\frac{e^2}{4\pi\varepsilon_0\varepsilon_r(f_1 l_0)} - \frac{e^2}{4\pi\varepsilon_0\varepsilon_r(f_2 l_0)} = \left(\frac{1}{f_1} - \frac{1}{f_2}\right)\frac{e^2}{4\pi\varepsilon_0\varepsilon_r l_0}$$

Therefore, $\Delta(B) \propto 1/l_0 \propto \sqrt{B}$, or $\alpha = 1/2$.

Grading: The key point here is to realize that the energy scale is determined by the Coulomb interaction, which scales inversely with a length scale (e.g., the magnetic length) that characterizes the mean electron distance (and its change). 1 point for the final expression and 1 point for writing down the correct relation between the length scale and the magnetic field.

(e) (**2 points**) As the magnetic field deviates from the exact filling $\nu = 1/m$ to a higher field, more vortices (whirlpools in the electron sea) are being created. They are not bound to electrons and behave like particles carrying effectively positive charges, hence known as quasiholes, compared to the negatively charged electrons. The amount of charge deficit in any of these quasiholes amounts to exactly $1/m$ of an electronic charge. An analogous argument can be made for magnetic fields slightly below ν and the creation of quasielectrons of negative charge $e^* = -e/m$. Assume the sample has an area A. At the quantized Hall plateau of $R_H = 3h/e^2$, calculate the amount of change in B that corresponds to the introduction of exactly one fractionally charged quasihole. (When their density is low, the quasiparticles are confined by the random potential generated by impurities and imperfections, hence the Hall resistance remains quantized for a finite range of B.)

Solution: The flux change due to the change of the magnetic field is

$$\Delta\phi = \Delta B(WL) = \frac{h}{e}$$

Therefore, $\Delta B = h/(eWL)$.

Grading: 2 points for the final expression.

(f) In Tsui *et al.* experiment,

- the magnetic field corresponding to the center of the quantized Hall plateau $R_H = 3h/e^2$, $B_{1/3} = 15$ Tesla,
- the effective mass of an electron in GaAs, $m^* = 0.067m_e$,
- the electron mass, $m_e = 9.1 \times 10^{-31}$ kg,
- Coulomb's constant, $k = 9.0 \times 10^9$ N·m^2/C^2,
- the vacuum permittivity, $\varepsilon_0 = 1/(4\pi k) = 8.854 \times 10^{-12}$ F/m,
- the relative permittivity (the ratio of the permittivity of a substance to the vacuum permittivity) of GaAs, $\varepsilon_r = 13$,
- the elementary charge, $e = 1.6 \times 10^{-19}$ C,
- Planck's constant, $h = 6.626 \times 10^{-34}$ J·s, and
- Boltzmann's constant, $k_B = 1.38 \times 10^{-23}$ J/K.

In our analysis, we have neglected several factors, whose corresponding energy scales, compared to $\Delta(B)$ discussed in (d), are either too large to excite or too small to be relevant.

(i) (**1 point**) Calculate the thermal energy E_{th} at temperature $T = 1.0$ K.

Solution: The thermal energy

$$E_{th} = k_BT = 1.38 \times 10^{-23} \times 1.0 = 1.38 \times 10^{-23} \text{ J}$$

Grading: 1 point for the numerical result.

(ii) (**2 point**) The electrons spatially confined in the whirlpools (or vortices) have a large kinetic energy. Using the uncertainty relation, estimate the order of magnitude of the kinetic energy. (This amount would also be the additional energy penalty if we put two electrons in the same whirlpool, instead of in two separate whirlpools, due to Pauli exclusion principle.)

Solution: The size of a vortex is of order

$$l_0 = \sqrt{\frac{h}{eB}} = \sqrt{\frac{6.626 \times 10^{-34}}{1.6 \times 10^{-19} \times 15}} = 1.66 \times 10^{-8}\mathrm{m}$$

According to the uncertainty relation, $p \sim \Delta p \sim h/l_0$. Therefore, the kinetic energy is

$$\begin{aligned}\frac{p^2}{2m^*} &= \frac{h^2}{2m^*}\frac{eB}{h} = \frac{h}{2}\frac{eB}{m^*} \\ &= \frac{6.626 \times 10^{-34} \times 1.6 \times 10^{-19} \times 15}{2 \times 0.067 \times 9.1 \times 10^{-31}} \\ &= 1.3 \times 10^{-20}\ \mathrm{J}\end{aligned}$$

Grading: 1 point for the final numerical result and 1 point for relating the characteristic length to the momentum through the uncertainty relation, and hence the kinetic energy. Note this is an estimate problem, hence any final numerical result within a factor of 2π can be regarded as correct.

(g) There are also a series of plateau at $R_H = h/ie^2$, where i = 1,2,3,... in Tsui *et al.* experiment, as shown in Figure 1(b). These plateaus, known as the integer quantum Hall effect (IQHE), were reported previously by K. von Klitzing in 1980. Repeating (c)-(f) for the integer plateaus, one realizes that the novelty of the FQHE lies critically in the existence of fractionally charged quasiparticles. R. de-Picciotto *et al.* and L. Saminadayar *et al.* independently reported the observation of fractional charges at the $\nu = 1/3$ filling in 1997. In the experiments, they measured the noise in the charge current across a narrow constriction, the so-called quantum point contact (QPC). In a simple statistical model, carriers with discrete charge e^* tunnel across the QPC and generate charge current I_B (on top of a trivial background). The number of the carriers n_τ arriving at the electrode during a sufficiently small time interval τ obeys Poisson probability distribution with parameter λ

$$P(n_\tau = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \tag{1}$$

where $k!$ is the factorial of k. You may need the following summation

$$e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}, \tag{2}$$

(i) (**2 point**) Determine the charge current I_B, which measures total charge per unit of time, in terms of λ and τ.

Solution: The current can be calculated by the ratio of the total charge carried by the averaged n_τ quasiparticles to the time interval τ.

$$\begin{aligned} \langle n_\tau \rangle = \sum_{k=1}^{\infty} kP(k) &= \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-1)!} \\ &= \lambda \sum_{k=0}^{\infty} P(k) \\ &= \lambda \end{aligned}$$

where we have used $\sum_k P(k) = 1$. Therefore,

$$I_B = \frac{\langle n_\tau \rangle e^*}{\tau} = \frac{\nu e \lambda}{\tau}$$

Grading: 1 point for correctly calculating the average charge under the Poisson distribution and 1 point for the final expression for the charge current.

(ii) (**2 points**) Current noise is defined as the charge fluctuations per unit of time. One can analyze the noise by measuring the mean square deviation of the number of current-carrying charges. Determine the current noise S_I due to the discreteness of the current-carrying charges in terms of λ and τ.

Solution: Similarly, the noise can be related to the averaged charge fluctuations during the time interval τ.

$$\begin{aligned}\left\langle (n_\tau - \langle n_\tau \rangle)^2 \right\rangle &= \langle n_\tau^2 \rangle - \langle n_\tau \rangle^2 \\ &= \sum_{k=1}^{\infty} k^2 P(k) - \lambda^2 \\ &= \left[\lambda^2 \sum_{k=0}^{\infty} P(k) + \sum_{k=1}^{\infty} kP(k) \right] - \lambda^2 \\ &= \lambda\end{aligned}$$

Therefore,

$$S_I = \frac{\left\langle (n_\tau - \langle n_\tau \rangle)^2 \right\rangle (e^*)^2}{\tau} = \frac{(\nu e)^2 \lambda}{\tau}$$

Grading: 1 point for correctly calculating the charge fluctuations under the Poisson distribution and 1 point for the final expression that relates the current noise to the charge fluctuations.

(iii) **(1 point)** Calculate the noise-to-current ratio S_I/I_B, which was verified by R. de-Picciotto *et al.* and L. Saminadayar *et al.* in 1997. (One year later, Tsui and Stormer shared the Nobel Prize in Physics with R. B. Laughlin, who proposed an elegant ansatz for the ground state wave function at $\nu = 1/3$.)

Solution: The noise-to-current ratio $S_I/I_B = e^* = \nu e$.

Grading: 1 point for the final expression.

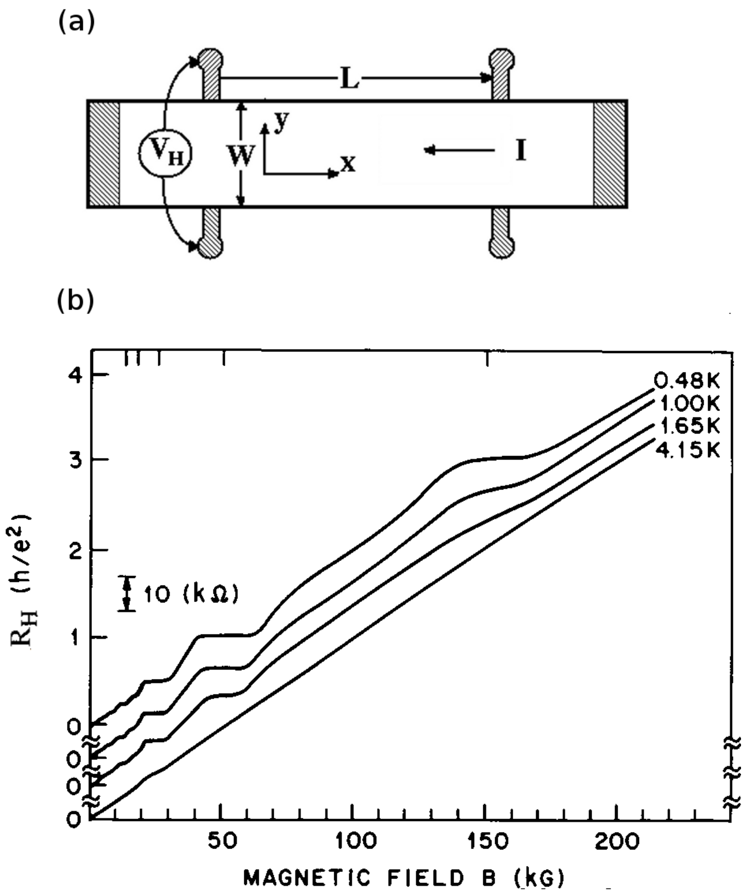


Figure 1: (a) Sketch of the experimental setup for the observation of the FQHE. As indicated, a current I is passing through a two-dimensional system in the longitudinal direction with an effective length L. The Hall voltage V_H is measured in the transverse direction with an effective width W. In addition, a uniform magnetic field B is applied perpendicular to the plane. The direction of the current is given for illustrative purpose only, which may not be correct. (b) Hall resistance R_H versus B at four different temperatures (curves shifted for clarity), adapted from the original publication on the FQHE. The features at $R_H = 3h/e^2$ are due to the FQHE.

Suggested Solutions for Question 2

Solution for (a):

（i）The electron motion equation

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}=-e\vec{v}\times\vec{B}\,. \qquad (1)\ (0.2\text{point})$$

Let $\vec{\omega}_c=-e\vec{B}/m$, we have

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}=-\vec{\omega}_c\times\vec{v}\,. \qquad (2)$$

Since $\vec{B}=B\hat{z}$, thus Equation (2) can be written to be

$$\begin{cases}\dfrac{\mathrm{d}v_x}{\mathrm{d}t}=\omega_c v_y\\ \dfrac{\mathrm{d}v_y}{\mathrm{d}t}=-\omega_c v_x\\ \dfrac{\mathrm{d}v_z}{\mathrm{d}t}=0\end{cases} \qquad (3)$$

The general solutions of Equation (3) are

$$\begin{cases}v_x=v_\perp\cos(\omega_c t+\varphi_y)\\ v_y=v_\perp\cos(\omega_c t+\varphi_y),\\ v_z=v_z(t=0)\end{cases} \qquad (4)\ (0.2\text{point})$$

where $\varphi_{x,y}$ are the initial phases. Due to $v_z(t=0)=0$, it is indicated that the motion of an electron remains to be perpendicular to the magnetic field. By further solving Equation (4), we obtain the motion trajectory of the electron,

$$\begin{cases}x=r_c\sin\omega_c t\\ y=r_c\cos\omega_c t\ ,\\ z=0\end{cases} \qquad (5)\ (0.2\text{point})$$

or

$$\begin{cases}x^2+y^2=r_c^{\,2}\\ z=0\end{cases}, \qquad (0.4\text{point})$$

where $|r_c|=\dfrac{v_\perp}{|\omega_c|}=\dfrac{mv_\perp}{eB}$ is the gyro radius.

（ii） The electric current generated by the circular motion of an electron are

$$I=\frac{e}{T}=\frac{e\omega_c}{2\pi}=\frac{e^2B}{2\pi m}。$$ （0.5point）

Based on the definition of the magnetic moment, we have

$$\vec{\mu}=I\vec{A}=-\frac{e^2\vec{B}}{2\pi m}\times\pi r_c^2=-\frac{e^2\vec{B}}{2\pi m}\times\frac{\pi m^2v_\perp^2}{e^2B^2}=-\frac{mv_\perp^2}{2B^2}\vec{B}。$$ （0.5point）

（iii）In the case, we have $v_z=v\cos\theta,\ v_\perp=v\sin\theta$. Since the velocity of the electron in z is not zero, the solution (5) in (i) becomes

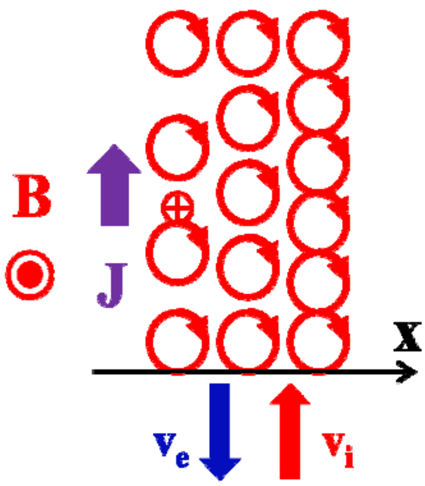
$$\begin{cases}x=r_c\sin\omega_c t\\ y=r_c\cos\omega_c t\\ z=vt\cos\theta\end{cases},\ \text{or}\ \begin{cases}x^2+y^2=r_c{}^2\\ z=vt\cos\theta\end{cases}.$$ (6)（0.4point）

The orbit equation (6) indicates that the electron has a spiral trajectory. The screw pitch is

$$h=v_zT=v\cos\theta\frac{2\pi}{\omega_c}=2\pi\frac{mv}{eB}\cos\theta.$$ （0.6point）

Solution for (b):

(i) Since the magnetic field and the plasma are uniform z, the orbits of ions and electrons can project into in the x-y plane. From the results of (a), we know that an ion has a left-hand circular motion and an electron has a right-hand circular motion. Due to the linear increase of the plasma density in x, the number of ions with upward motion is less than that with downward motion at a given x position, which leads a net upward ion flow. Similarly, electrons have a net downward flow. Combining the ion and election flows, we have a net upward electric current as illustrated below in schematic drawing.



（2.0points）

(ii) Based on $\vec{\mu}=-\frac{m{v_\perp}^2}{2B^2}\vec{B}$ from Problem (a), the total magnetic moments per unit volume (i.e., the magnetization) for ions and electrons are

$$
\begin{aligned}
M_{i,e} &= \iiint f_{i,e}(x,v)\mu_{i,e}\mathrm{d}^3\vec{v} \\
&= -\int_0^{\infty}\int_{-\infty}^{\infty} n(x)\left(\frac{m_{i,e}}{2\pi kT}\right)^{3/2} e^{-m_{i,e}(v_\perp^2+v_\parallel^2)/2kT}\frac{m_{i,e}{v_\perp}^2}{2B}2\pi v_\perp \mathrm{d}v_\perp \mathrm{d}v_\parallel \\
&= -n(x)\int_0^{\infty}\frac{m_{i,e}}{2\pi kT}e^{-m_{i,e}v_\perp^2/2kT}\frac{m_{i,e}{v_\perp}^2}{2B}2\pi v_\perp \mathrm{d}v_\perp \mathrm{d}v_\parallel \int_{-\infty}^{\infty}\left(\frac{m_{i,e}}{2\pi kT}\right)^{1/2}e^{-m_{i,e}v_\parallel^2/2kT}\mathrm{d}v_\parallel \\
&= -\frac{kT}{B}n(x)
\end{aligned}
$$

（1.5point）

The total magnetization for a plasma：

$$M=M_i+M_e=-\frac{2kT}{B}n(x)=-\frac{p(x)}{B}$$

Therefore, we have

$$\beta=-2\,.$$ （0.5point）

Solution for (c)

Using the equation for the magnetization field $B_{mx}=\mu_o M$ and the x component of the Earth's dipole magnetic field at (x=10R$_E$，y=0, z=1R$_E$), we have

$$\left|B_{mx}/B_d\right|=\mu_0\frac{p(z)}{B_d^2(10R_E,0,z)}\Big|_{z=R_E}=\frac{\mu_0 e^{-0.25}P_0}{28.5\times10^{-8}B_0^2}\approx1.0\,.$$ （1.0point）

Solution for (d)

（i）Under the cylindrical coordinate system, the motion of an electron can project into the r and θ plane，the Lorentz force is $F_z=ev_\theta B_r$.

By taking a gyro average, the averaged Lorentz force becomes

$$\langle F_z\rangle=\frac{1}{2}ev_\perp r_c\frac{dB}{dz}=-\mu\frac{dB}{ds}\,.$$ （1.0point）

Since $\frac{dB}{dz}=\frac{dB}{ds}$, we have

$$\langle F_\parallel\rangle=\langle F_z\rangle=-\mu\frac{dB}{ds}\,.$$

The acceleration of the guiding center of the electron can be obtained by the Newton's law, i.e.,

$$m\frac{\mathrm{d}v_{//}}{\mathrm{d}t} = -\mu\frac{dB}{ds}, \qquad (7) \qquad (0.5\text{point})$$

where $v_{//} = \mathrm{d}s/\mathrm{d}t$. Equation (7) can be written to be

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}mv_{//}^2) = -\mu\frac{\mathrm{d}B}{\mathrm{d}t}. \qquad (8)$$

Since the total kinetic energy is conserved, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}mv_{//}^2 + \frac{1}{2}mv_{\perp}^2) = 0$, (0.5point)

we can obtain

$$\mu\frac{\mathrm{d}B}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}mv_{//}^2) = \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}mv_{\perp}^2) = \frac{\mathrm{d}}{\mathrm{d}t}(\mu B) = \mu\frac{\mathrm{d}B}{\mathrm{d}t} + B\frac{\mathrm{d}\mu}{\mathrm{d}t}, \qquad (9) \qquad (1.0\text{point})$$

Combining Equations (8) and (9), we get

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0. \qquad (10)$$

（ii）From the motion constant of the magnetic moment of an electron, the perpendicular velocity increases with increase of the magnetic field, which means the parallel velocity decreases due to the conservation of the total kinetic energy. When the electron arrives at the point "P_3", its parallel velocity decreases to zero, then the electron will not escape from the magnetic mirror field. Thus, the initial velocityshould be

$$\frac{v_{\perp 0}^2}{B_0} \geq \frac{v^2}{B_m}. \qquad (0.5\text{point})$$

Since $v_{\perp 0} = v\sin\theta$, we obtain

$$\theta_{cr} \geq \arcsin\left(\sqrt{\frac{B_0}{B_m}}\right), \qquad (0.5\text{point})$$

i.e., this is the condition for the electron confined in the magnetic mirror field.

Solution for (e)

Since the guiding center of the electron motion is always confined in the y=0 plane, the Earth's dipole magnetic field $\vec{B} = \frac{B_0 R_E^3}{r^5}(-3xz\hat{x} + (x^2 - 2z^2)\hat{z})$. At the initial position $(6R_E, 0, 0)$ of an electron, the magnetic field strength

$$B_i = \frac{B_0 R_E^3}{r^5}(x^2) = \frac{B_0}{216}.$$ （0.2point）

At the altitude H=200km and the latitude $\theta_L = 60°$, thus

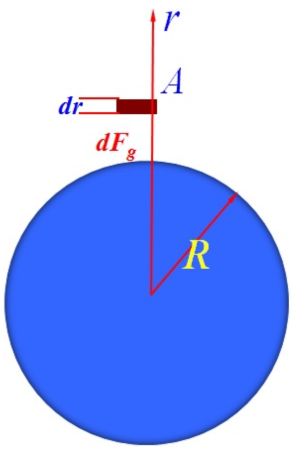
$$B_L = (B_x^2 + B_z^2)^{1/2} = \frac{\sqrt{13}}{2} B_0.$$ （0.2point）

With the same argument in (d) (ii), the condition for the electron to arrive in below the altitude H=200km with the latitude $\theta_L = 60°$ is

$$\theta < 0.05 = 2.8°.$$ （0.6point）

Solution for (f)

(i)



Based on the balance between the gravitational force and the force from the difference of the atmospheric pressure, we obtain

$$A[p(r+dr) - p(r)] = Adp = -\frac{GM(\rho A dr)}{r^2}$$

or

$$dp = -\frac{GM(\rho dr)}{r^2}.$$ （11） （0.5point）

For the ideal gas, we have

$$p = \frac{NkT}{V} = \frac{NmkT}{Vm} = \frac{\rho kT}{m}.$$ （12）

Since the atmospheric temperature is constant, Equation (12) can written to be

$$p = \frac{p_0}{\rho_0}\rho \,. \qquad (13) \qquad (0.5\text{point})$$

Inserting Equation (13) into (11), we have

$$\frac{d\rho}{\rho} = -\frac{\rho_0 GMdr}{p_0}\,. \qquad (14)$$

By integrating Equation (14), we obtain

$$\rho = \rho_0 e^{\frac{\rho_0 GM}{P_0}(\frac{1}{r}-\frac{1}{R_E})} = \rho_0 e^{-\frac{\rho_0 g R_E}{P_0}(1-\frac{R_E}{r})} \approx \rho_0 e^{-\frac{\rho_0 g H}{P_0}}\,. \qquad (15) \qquad (0.5\text{point})$$

Thus, the ration of the atmospheric density at the altitudes H=160km and H=220km is

$$\frac{\rho(H_1 = 160km)}{\rho(H_2 = 220km)} \approx e^{\frac{\rho_0 g(H_2 - H_1)}{P_0}} \approx 2.44\times 10^3\,. \qquad (16) \qquad (0.5\text{point})$$

（ii）At the altitude H=160km，the collision frequency of atmosphericmolecules is

$$\nu = \frac{\nu_0\rho}{\rho_0} = \nu_0 e^{-\frac{\rho_0 g H}{P_0}} \approx 0.93/s\,. \qquad (17) \qquad (0.5\text{point})$$

Since the lifetime of the oxygen atom in the first excited state lasts about 110s, the oxygen atom will collide with other molecules over one hundred times. Thus, the high-frequent collision between particles can de-excite oxygen atoms before they have a chance to radiate. But, the oxygen atoms in the second excited state will emit photons since their lifetime is so short. Thus, aurora at H=160km is resulted from emission of the oxygen atoms in the second excited state. The wavelength is

$$\lambda = \frac{hc}{e\Delta V} = \frac{(6.626\times 10^{-34} J\cdot s)(2.998\times 10^8 m/s)}{(1.6\times 10^{-19} J/eV)(2.21eV)} = 562nm\,. \qquad (0.8\text{point})$$

The color for this wavelength is green. （0.2point）

At the altitude H=220km，the collision frequency of atmosphericmolecules is

$$\nu = \frac{\nu_0\rho}{\rho_0} = \nu_0 e^{-\frac{\rho_0 g H}{P_0}} \approx 3.8\times 10^{-4}/s\,. \qquad (18) \qquad (0.5\text{point})$$

All oxygen atoms in the first and second excited states have a chance to radiate since the collision frequency is so low. Because the number of the oxygen atoms in the first excited state is much larger than in the second excited state, we observe the aurora color is from the emission of the oxygen atoms in the first excited state. The wavelength is

$$\lambda = \frac{hc}{e\Delta V} = \frac{(6.626\times 10^{-34} J\cdot s)(2.998\times 10^8 m/s)}{(1.6\times 10^{-19} J/eV)(1.96eV)} = 633nm\,. \qquad (0.8\text{point})$$

The color for this wavelength is red. （0.2point）

Solution for (g)

At the geosynchronous orbit, we have

$$G\frac{mM}{(h+R_E)^2}=m\frac{v^2}{h+R_E}=m(h+R_E)\omega^2.$$

or

$$(h+R_E)^3=\frac{GM}{\omega^2}=\frac{gR_E^2}{\omega^2}=\frac{g}{\omega^2 R_E}R_E^3.$$

Using $\omega=2\pi/(24\times3600s)$, we have

$$h=5.6R_E.$$ （0.5point）

Using $\Delta w=F\Delta x=\Delta E_k$, we have

$$(f_m-f_s)\Delta s\Delta x=\frac{1}{2}\Delta m v_s^2.$$

or

$$\frac{1}{2}\rho v_s^2+\frac{B_s^2}{2\mu_0}=\frac{B_m^2}{2\mu_0}. \quad (19)$$ （1.0point）

where ρ is the mass density. At the position $x=R_E+h=6.6R_E, y=z=0$, the Earth's dipole magnetic field strength is

$$B_d=\frac{B_0}{290}\sim 100nT$$

From Equation (19), we obtain

$$v=330km/s.$$ （0.5point）

With this speed of the solar wind, the position of the dayside magetopause is 6.6R_E at the geosynchronous orbit. Thus, a geosynchronous satellite could be damaged by the solar wind.

Solution for Question 3

Figure 1 shows a Fabry-Perot (F-P) etalon, in which air pressure is tunable. The F-P etalon consists of two glass plates with high-reflectivity inner surfaces. The two plates form a cavity in which light can be reflected back and forth. The outer surfaces of the plates are generally not parallel to the inner ones and do not affect the back-and-forth reflection. The air density in the etalon can be controlled. Light from a Sodium lamp is collimated by the lens L1 and then passes through the F-P etalon. The transmitivity of the etalon is given by $T = \dfrac{1}{1 + F\sin^2(\delta/2)}$, where $F = \dfrac{4R}{(1-R)^2}$, R is the reflectivity of the inner surfaces, $\delta = \dfrac{4\pi nt\cos\theta}{\lambda}$ is the phase shift of two neighboring rays, n is the refractive index of the gas, t is the spacing of inner surfaces, θ is the incident angle, and λ is the light wavelength.

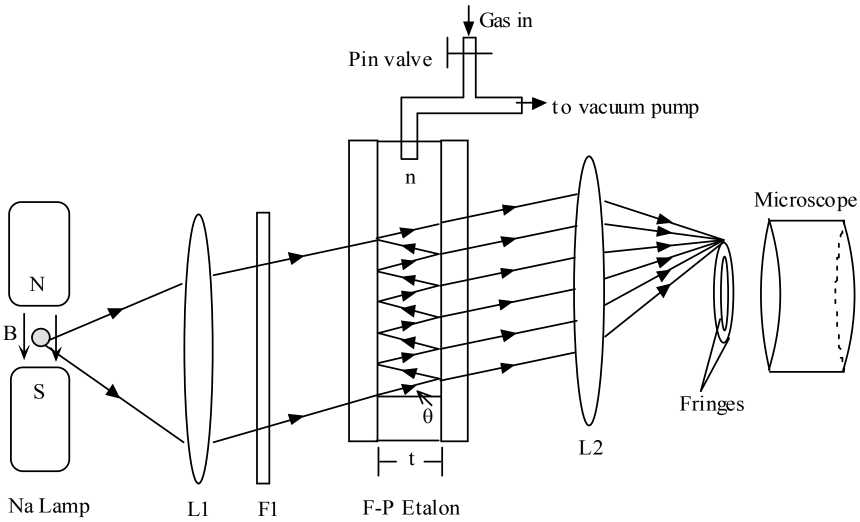


Figure 1

The Sodium lamp emits D1 ($\lambda = 589.6nm$) and D2 ($589nm$) spectral lines and is located in a tunable uniform magnetic field. For simplicity, an optical filter F1 is assumed to only allow the D1 line to pass through. The D1 line is then collimated to the F-P etalon by the lens L1. Circular interference fringes will be present on the focal plane of the lens L2 with a focal length f=30cm. Different fringes have the different incident angle θ. A microscope is used to observe the fringes. We take the reflectivity R= 90% and the inner-surface spacing t=1cm.

Some useful constants : $h = 6.626\times10^{-34}\,J\cdot s$, $e = 1.6\times10^{-19}\,C$, $m_e = 9.1\times10^{-31}\,kg$, $c = 3.0\times10^{8}\,ms^{-1}$.

Solution for Question 3

（a） **（3points）** The D1 line ($\lambda = 589.6nm$) is collimated to the F-P etalon. For the vacuum case (n=1.0), please calculate (i) interference orders m_i, (ii) incidence angle θ_i and (iii) diameter D_i for the first three（i=1，2，3）fringes from the center of the ring patterns on the focal plane.

Solution：

The transmittivity of the F-P etalon is given by：

$$T = \frac{1}{1 + F\sin^2\frac{\delta}{2}}$$

For bright fringes, we have

$$T = 1 \text{ i.e. } \sin^2\frac{\delta}{2} = 0$$

$$\frac{\delta}{2} = m\pi$$

$$2nt\cos\theta = m\lambda$$

For n=1.0，t=1cm，$\lambda = 589.6nm$，thus:

$$\cos\theta_i = \frac{m_i}{2nt/\lambda} = \frac{m_i}{33921.3}$$ (a1)（**1 point if Eqs. (a2-a3) are not correct.**）

Because of $\cos\theta \le 1$，so the orders of the first three fringes are:

$$m_1 = 33921, m_2 = 33920, m_3 = 33919$$ **(a2)（1 point）**

The incident angles of the first three fringes are：

$$\theta_1 = 0.241^0, \theta_2 = 0.502^0, \theta_3 = 0.667^0$$ **(a3)（1 point）**

The fringe diameter is given by：

$$D_i = 2f\tan\theta_i \approx 2f\theta_i$$ (a4)（**0.5 point if Eq(a5) is not correct.**）

For the focal length f=30cm，thus:

$$D_1 = 2.52mm, D_2 = 5.26mm, D_3 = 6.99mm$$ **(a5)（1 point）**

Solution for Question 3

(b)(**3 points**) As shown in Fig. 2, the width ε of the spectral line is defined as the full width of half maximum (FWHM) of light transmitivity T regarding the phase shift δ. The resolution of the F-P etalon is defined as follows: for two wavelengths λ and $\lambda+\Delta\lambda$, when the central phase difference $\Delta\delta$ of both spectral lines is larger than ε, they are thought to be resolvable; then the etalon resolution is $\lambda/\Delta\lambda$ when $\Delta\delta=\varepsilon$. For the vacuum case, the D1 line ($\lambda=589.6nm$), and because of the incident angle $\theta\approx 0$, take $\cos\theta\approx 1.0$, please calculate:

(i) the width ε of the spectral line.

(ii) the resolution $\lambda/\Delta\lambda$ of the etalon.

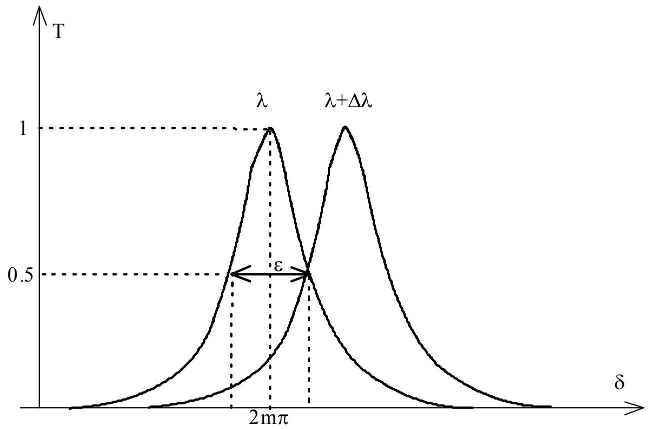


Figure 2

Solution:

The half maximum occurs at:

$$\delta = 2m\pi \pm \frac{\varepsilon}{2}$$ (b1) (**0.2 point if Eq.(b3) is wrong.**)

Given that $T=0.5$, thus:

$$F\sin^2\frac{\delta}{2}=1$$ (b2) (**0.2 point if Eq.(b3) is wrong.**)

$$\varepsilon=\frac{4}{\sqrt{F}}=\frac{2(1-R)}{\sqrt{R}}=\frac{2(1-0.9)}{\sqrt{0.9}}=0.21\,\text{rad}\,(\text{or}\,12.03\text{degree})$$ (b3) (**1 point**)

The phase shift δ is given by:

$$\delta=\frac{4\pi nt\cos\theta}{\lambda}$$

For a small $\Delta\lambda$, thus:

$$\Delta\delta=-\frac{4\pi nt\cos\theta}{\lambda^2}\Delta\lambda$$ **(b4)** (**1 point if Eq. (b5) is wrong.**)

For $\Delta\delta=\varepsilon$ and $\lambda=589.6nm$, we get:

$$\frac{\lambda}{\Delta\lambda}=\frac{\pi nt\sqrt{F}\cos\theta}{\lambda}=\frac{3.14\times1.0\times1.0\times10^{-2}\times\sqrt{360}\times1.0}{589.6\times10^{-9}}=1.01\times10^{6}$$ (b5) (**2 points**)

(**1.5 point if the final value of Eq. (b5) is wrong.**)

Solution for Question 3

（c）（**1 point**） As shown in Fig. 1, the initial air pressure is zero. By slowly tuning the pin valve, air is gradually injected into the F-P etalon and finally the air pressure reaches the standard atmospheric pressure. On the same time, ten new fringes are observed to produce from the center of the ring patterns on the focal plane. Based on this phenomenon, calculate the refractive index of air n_{air} at the standard atmospheric pressure.

Solution：
From Question (a), we know that the order of the 1st fringe near the center of ring patterns is m=33921 at the vacuum case (n=1.0). When the air pressure reaches the standard atmospheric pressure, the order of the 1st fringe becomes m+10, so we have:

$$n_{air} = \frac{m+10}{2t/\lambda} = \frac{33931}{33921} = 1.00029$$. (c1)（**1 point**）

（**0.2 point for appearing the term of (m+10) when the final value of Eq.(c1) is wrong.**
Or
0.8 point for the correct final expression (including other correct forms) without the correct value.）

Solution for Question 3

（d） **(2 points)** Energy levels splitting of Sodium atoms occurs when they are placed in a magnetic field. This is called as the Zeeman effect. The energy shift given by $\Delta E = m_j g_k \mu_B B$, where the quantum number m_j can be J，J-1，…，-J+1，-J，J is the total angular quantum number, g_k is the Landé factor, $\mu_B = \frac{he}{4\pi m_e}$ is Bohr magneton，h is the Plank constant，e is the electron charge，m_e is the electron mass, B is the magnetic field. As shown in Fig. 3, the D1 spectral line is emitted when Sodium atoms jump from the energy level $^2P_{1/2}$ down to $^2S_{1/2}$. We have $J = \frac{1}{2}$ for both $^2P_{1/2}$ and $^2S_{1/2}$. Therefore, in the magnetic field, each energy level will be split into two levels. We define the energy gap of two splitting levels as ΔE_1 for $^2P_{1/2}$ and ΔE_2 for $^2S_{1/2}$ respectively ($\Delta E_1 < \Delta E_2$). As a result, the D1 line is split into 4 spectral lines (a, b, c, and d), as showed in Fig. 3. Please write down the expression of the frequency (ν) of four lines a, b, c, and d.

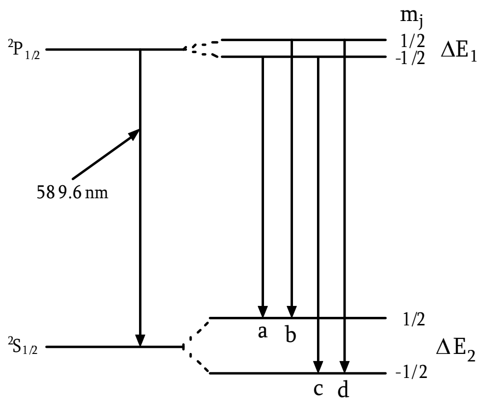


Figure 3

Solution：

The frequency of D1 line ($^2P_{1/2}$ to $^2S_{1/2}$) is given by： $\nu_0 = c/\lambda \ (\lambda = 589.6nm)$

When magnetic field B is applied，the frequency of the line a,b,c,d are expressed as:

1) $^2P_{1/2}$ (mj=-1/2) → $^2S_{1/2}$ (mj =1/2): frequency of (a）): $\nu_a = \nu_0 - \frac{1}{2h}(\Delta E_1 + \Delta E_2)$；**（0.5 point）**

2) $^2P_{1/2}$ (mj=1/2) → $^2S_{1/2}$ (mj=1/2): frequency of（b）: $\nu_b = \nu_0 - \frac{1}{2h}(\Delta E_2 - \Delta E_1)$；**（0.5 point）**

3) $^2P_{1/2}$ (mj=-1/2) → $^2S_{1/2}$ (mj=-1/2): frequency of（c）: $\nu_c = \nu_0 + \frac{1}{2h}(\Delta E_2 - \Delta E_1)$；**（0.5 point）**

4) $^2P_{1/2}$ (mj=1/2) → $^2S_{1/2}$ (mj=-1/2): frequency of（d）: $\nu_d = \nu_0 + \frac{1}{2h}(\Delta E_1 + \Delta E_2)$；**（0.5 point）**

（The results maybe have other correct forms.
But, 0.4 point for each result without the coefficient of $1/2$.）

Solution for Question 3

(e) **(3 points)** As shown in Fig. 4, when the magnetic field is turned on, each fringe of the D1 line will split into four sub-fringes (1, 2, 3, and 4). The diameter of the four sub-fringes near the center is measured as D_1, D_2, D_3, and D_4. Please give the expression of the splitting energy gap ΔE_1 of $^2P_{1/2}$ and ΔE_2 of $^2S_{1/2}$.

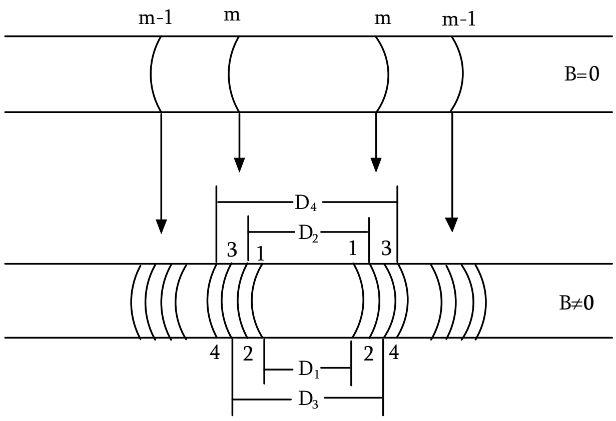


Figure 4

Solution: $\theta_m << 1, \cos\theta_m = 1 - \frac{\theta_m^2}{2}$, (e1)（**0.2point if Eq. (e4) is wrong.**）

$2nt\cos\theta_m = m\lambda$, $1 - \frac{\theta_m^2}{2} = \frac{m\lambda}{2nt}$, (e2)（**0.2point if Eq. (e4) is wrong.**）

$\lambda \to \lambda + \Delta\lambda, \theta_m \to \theta_m'$,

$1 - \frac{\theta_m'^2}{2} = \frac{m(\lambda + \Delta\lambda)}{2nt}$,

$\frac{\theta_m^2 - \theta_m'^2}{2} = \frac{m\Delta\lambda}{2nt}$ (e3)（**0.2point if Eq. (e4) is wrong.**）

$2f\theta_m = D_m$, $\frac{D_m^2 - D_m'^2}{8f^2} = \frac{m\Delta\lambda}{2nt} = \frac{\Delta\lambda}{\lambda}$

$\Delta\lambda = \lambda\frac{D_m^2 - D_m'^2}{8f^2}$ (e4)（**1 point**）

The lines a, b, c, and d correspond to sub-fringe 1, 2, 3, and 4. From Question (d), we have.

The wavelength difference of the spectral line a and b is given by:

$$\Delta\lambda_1 = \lambda\frac{D_2^2 - D_1^2}{8f^2}$$

Solution for Question 3

$\Delta E_1 = h(\nu_b - \nu_a)$, $\Delta E_2 = h(\nu_d - \nu_b)$

or $\Delta E_1 = h(\nu_d - \nu_c)$, $\Delta E_2 = h(\nu_c - \nu_a)$ (e5)

（**0.5 point for each subequation in Eq (e5) if Eqs. (e6) and (e7) are totally wrong.**）

The wavelength difference of the spectral line a and b is given by:

$$\Delta\lambda_1 = \lambda \frac{D_2^2 - D_1^2}{8f^2}$$

Then we obtain

$$\Delta E_1 = h\Delta\nu_1 = \left| -\frac{hc}{\lambda} \bullet \frac{D_2^2 - D_1^2}{8f^2} \right| = \frac{hc}{\lambda} \bullet \frac{D_2^2 - D_1^2}{8f^2}$$

（or $\Delta E_1 = h\Delta\nu_1 = \left| -\frac{hc}{\lambda} \bullet \frac{D_4^2 - D_3^2}{8f^2} \right| = \frac{hc}{\lambda} \bullet \frac{D_4^2 - D_3^2}{8f^2}$ ） (e6)（**1 point**）

Similarly, for ΔE_2，we get

$$\Delta\lambda_2 = \lambda \frac{D_4^2 - D_2^2}{8f^2}$$

$$\Delta E_2 = h\Delta\nu_2 = \left| -\frac{hc}{\lambda} \bullet \frac{D_4^2 - D_2^2}{8f^2} \right| = \frac{hc}{\lambda} \bullet \frac{D_4^2 - D_2^2}{8f^2}$$

（or $\Delta E_1 = h\Delta\nu_1 = \left| -\frac{hc}{\lambda} \bullet \frac{D_3^2 - D_1^2}{8f^2} \right| = \frac{hc}{\lambda} \bullet \frac{D_3^2 - D_1^2}{8f^2}$ ） (e7)（**1 point**）

(Eqs (e6 and e7) have other correct forms which should be in terms of D_1， D_2， D_3， and D_4)

(2.5 points for the final expressions only with the incorrect coefficients.)

Solution for Question 3

(f) (**3 points**) For the magnetic field B=0.1T, the diameter of four sub-fringes is measured as: $D_1 = 3.88mm$, $D_2 = 4.05mm$, $D_3 = 4.35mm$, and $D_4 = 4.51mm$. Please calculate the Landé factor g_{k1} of $^2P_{1/2}$ and g_{k2} of $^2S_{1/2}$.

Solution:

Given that B=0.1T, so we have:

$$\mu_B B = \frac{heB}{4\pi m_e} = \frac{6.626\times10^{-34}\times0.1}{4\times3.14\times9.1\times10^{-31}} = 5.79\times10^{-6}eV$$ (f1) (**0.2point if Eq. (f4) is wrong.**)

$$\Delta E_1 = g_{k1}\mu_b B = \frac{hc}{\lambda}\bullet\frac{D_2^2 - D_1^2}{8f^2};$$ (f2)

(or, $\Delta E_1 = g_{k1}\mu_b B = \frac{hc}{\lambda}\bullet\frac{D_4^2 - D_3^2}{8f^2}$) (**0.5point if Eq. (f4) is wrong.**)

For the D1 spectral line, $\lambda = 589.6nm$, so we can get:

$$\frac{hc}{\lambda} = \frac{6.626\times10^{-34}\times3\times10^8}{5.896\times10^{-7}\times1.6\times10^{-19}} = 2.11eV,$$ (f3) (**0.2point if Eq. (f4) is wrong.**)

thus:

$$g_{k1} = \frac{2.11}{5.79\times10^{-6}}\bullet\frac{D_2^2 - D_1^2}{8f^2} = \frac{2.11}{5.79\times10^{-6}}\bullet\frac{\left(4.05\times10^{-3}\right)^2 - \left(3.88\times10^{-3}\right)^2}{8\times0.3\times0.3} = 0.68;$$

(**1.5 points**)

(or $g_{k1} = \frac{2.11}{5.79\times10^{-6}}\bullet\frac{D_4^2 - D_3^2}{8f^2} = \frac{2.11}{5.79\times10^{-6}}\bullet\frac{\left(4.51\times10^{-3}\right)^2 - \left(4.35\times10^{-3}\right)^2}{8\times0.3\times0.3} = 0.72$)

Similarly, we get:

$$g_{k2} = \frac{2.11}{5.79\times10^{-6}}\bullet\frac{D_4^2 - D_2^2}{8f^2} = \frac{2.11}{5.79\times10^{-6}}\bullet\frac{\left(4.51\times10^{-3}\right)^2 - \left(4.05\times10^{-3}\right)^2}{8\times0.3\times0.3} = 1.99$$ (**1.5 points**)

(or $g_{k2} = \frac{2.11}{5.79\times10^{-6}}\bullet\frac{D_3^2 - D_1^2}{8f^2} = \frac{2.11}{5.79\times10^{-6}}\bullet\frac{\left(4.35\times10^{-3}\right)^2 - \left(3.88\times10^{-3}\right)^2}{8\times0.3\times0.3} = 1.95$)

(2 points for the correct final expressions if the final values are wrong.)

(***Comment: the theory value of g_{k1} and g_{k2} is 2/3 and 2**)

Solution for Question 3

（g） **(2 points)** The magnetic field on the sun can be determined by measuring the Zeeman effect of the Sodium D1 line on some special regions of the sun. One observes that, in the four split lines, the wavelength difference between the shortest and longest wavelength is 0.012nm by a solar spectrograph. What is the magnetic field B in this region of the sun?

Solution：

We have $\Delta E_1 = g_{k1}\mu_B B$ and $\Delta E_2 = g_{k2}\mu_B B$;

The line a has the longest wavelength and the line d has the shortest wavelength line. The energy difference of the line a and d is

$$\Delta E = \Delta E_1 + \Delta E_2 = (g_{k1} + g_{k2})\mu_B B .$$ **(g1)** （**0.5point if Eq. (g3) is wrong.**））

$$\Delta \nu = \left| -\frac{c\Delta\lambda}{\lambda^2} \right| = \frac{c\Delta\lambda}{\lambda^2}$$ (g2)（**0.5 point**）

$$\Delta \nu = \frac{(g_{k1} + g_{k2})\mu_B B}{h}$$ （**g3**）（**0.5 point**）

$$\mu_B = \frac{he}{4\pi m_e}$$

So the magnetic field B is given by:

$$\begin{aligned} B &= \frac{4\pi m_e \Delta\lambda c}{\lambda^2 (g_{k1} + g_{k2}) e} \\ &= \frac{4\times 3.14\times 9.1\times 10^{-31}\times 0.012\times 10^{-9}\times 3\times 10^{8}}{(589.6\times 10^{-9})^2\times 2.67\times 1.6\times 10^{-19}} T \\ &= 0.2772T \\ &= 2772.1 Gauss \end{aligned}$$ (g4)（**1 point**）

（**0.5 point if the first line in Eq (g4) is correct.**）

Solution for Question 3

(h)（**3 points**） A Light- Emitting Diode (LED) source with a central wavelength $\lambda = 650nm$ and spectral width $\Delta\lambda = 20nm$ is normally incident ($\theta = 0$) into the F-P etalon shown in Fig. 1. For the vacuum case, find (i) the number of lines in transmitted spectrum and (ii) the frequency width $\Delta\nu$ of each line?

Solution：

The wavelength of transmitted spectral lines is given by:

$$2nt = m\lambda_m$$ (h1)（**0.5 point if Eq. (h2) is wrong.**）

$$\nu_m = \frac{c}{\lambda_m}$$

$$\nu_m = \frac{mc}{2nt}$$

$$\Delta\nu_m = \frac{c}{2nt} = 1.5\times10^{10}\,Hz$$ (h2)（**1 point**）

The frequency width of the input LED is：

$$\Delta\nu_s = \left|-\frac{c\Delta\lambda}{\lambda^2}\right|$$
$$= \frac{3\times10^8\times20\times10^{-9}}{\left(650\times10^{-9}\right)^2} = 1.42\times10^{13}\,Hz$$ (h3)

（**0.5point if the first line in Eq. (h3) is correct.**）

So we have the number of transmitted spectral line：

$$N = \frac{\Delta\nu_s}{\Delta\nu_m}$$
$$= \frac{1.42\times10^{13}}{1.5\times10^{10}} = 946$$ (h4)（**1 point**）

（**0.5point if the first line in Eq. (h4) is correct.**）

The spectral width of transmitted spectral line is $\Delta\lambda = \frac{\lambda^2}{\pi nt\sqrt{F}}$, then we have

$$\Delta\nu = \frac{c}{\pi nt\sqrt{F}}$$
$$= \frac{3\times10^8}{3.14\times1.0\times10\times10^{-3}\times\sqrt{360}} = 5.0\times10^8\,Hz$$ (h5)（**1 point**）

（**0.5point if the first line in Eq. (h5) is correct.**）