General instructions: Theoretical Examination (30 points)

Failure to comply with any of the following conditions may result in disqualification

July 21, 2025

The theoretical examination lasts for 5 hours and is worth a total of 30 points.

The beginning and end of the examination will be indicated by an announcement, as well as every hour indicating the elapsed time, and thirty, fifteen and five minutes before the end of the examination.

Do not open the envelopes until instructed to do so.

The following items are provided for your use on the table: (1) a ballpoint pen, (2) a mechanical pencil, (3) an eraser, (4) a graduated ruler, (5) a compass, and (6) a scientific calculator.

During the exam

- Use the ballpoint pen provided. If you use the mechanical pencil to draft your notes, figures, tables and graphs, you must trace the outlines of the final version with the ballpoint pen.
- Use the dedicated sheets labeled **A** for your final answers. Fill in appropriate sections with your answers and necessary observations. Draw graphs as required. Cross out anything you do not want marked.
- Blank working sheets labeled **W** are provided for drafting. Use the designated ones. Cross out any unneeded answers and rough work that do not need to be graded. Use only the front face of each sheet and keep the margin outside the border clean.
- Additional working sheets labeled **Z** are available upon request. Raise the "Help" flag and let the Invigilator know.
- Keep your answers concise and legible. Use equations, logical operators, symbols, and sketches that best convey your thoughts. Avoid being lengthy and wordy as the markers may not be multilingual.
- Uncertainty quantification is not required unless otherwise specified.
- Do not leave your booth without permission. If you need a washroom break or any other assistance, raise the flag(s) marked "Toilet", "Water", or "Help".

At the end of the exam

- Stop writing immediately when the end of the exam is announced.
- Put all the sheets in the windowed envelope. Organize them faceup in the following order : cover sheet on top, General Instructions, question sheets (**Q**), answer sheets (**A**), working sheets (**W**), and additional working sheets (**Z**), if any. Arrange them according to their page numbers. In the final check, make sure that your ID, name, and seat number on the cover sheet are visible through the window.
- Your Invigilator will let you know when you can leave. You may take the remaining items with you, for example, the ballpoint pen, the mechanical pencil, the graduated ruler, the eraser, the compass, the scientific calculator, the bottle of drinking water, and snacks.

Specific instructions

Throughout the examination, always refer to the quantities indicated in the instructions, unless otherwise stated. In all cases, the units of all quantities used should be given.

General Data Sheet

Constant	Notation	Value	Unit
Speed of light in vacuum	c	$2.997\,924\,58 \times 10^{8}$	$\mathrm{m \cdot s^{-1}}$
Planck constant	h	$6.626\,070\,15 \times 10^{-34}$	$\mathrm{kg \cdot m^2 \cdot s^{-1}}$
Reduced Planck constant	$\hbar = \frac{h}{2\pi}$	$1.054\,571\,818 \times 10^{-34}$	$\mathrm{kg \cdot m^2 \cdot s^{-1}}$
Boltzmann constant	k_{B}	$1.380\,649 \times 10^{-23}$	$\mathrm{kg \cdot m^2 \cdot s^{-2} \cdot K^{-1}}$
Avogadro constant	N_{A}	$6.022\,140\,76 \times 10^{23}$	$\mathrm{mol^{-1}}$
Molar gas constant	R	$8.314\,462\,618\,153\,24$	$\mathrm{kg \cdot m^2 \cdot s^{-2} \cdot mol^{-1} \cdot K^{-1}}$
Elementary charge	e	$1.602\,176\,634 \times 10^{-19}$	$\mathrm{A \cdot s}$
Universal constant of gravitation	G	$6.674\,30(15) \times 10^{-11}$	$\mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$
Standard acceleration due to gravity	g	$9.806\,65$	$\mathrm{m \cdot s^{-2}}$
Stefan Boltzmann constant	σ	$5.670\,374\,419 \times 10^{-8}$	$\mathrm{kg \cdot s^{-3} \cdot K^{-4}}$
Vacuum permeability (magnetic constant)	μ_0	$1.256\,637\,061\,27(20) \times 10^{-6}$	$\mathrm{kg \cdot m \cdot A^{-2} \cdot s^{-2}}$
Vacuum permittivity (electrical constant)	ϵ_0	$8.854\,187\,818\,8(14) \times 10^{-12}$	$\mathrm{A^2 \cdot s^4 \cdot kg^{-1} \cdot m^{-3}}$
Rydberg constant	R_{∞}	$1.097\,373\,156\,815\,7(12) \times 10^{7}$	$\mathrm{m^{-1}}$
Mass of the electron	m_e	$9.109\,383\,713\,9(28) \times 10^{-31}$	kg
Mass of the proton	m_p	$1.672\,621\,925\,95(52) \times 10^{-27}$	kg
Mass of the neutron	m_n	$1.674\,927\,500\,56(85) \times 10^{-27}$	kg
Atomic mass constant	m_u	$1.660\,539\,068\,92(52) \times 10^{-27}$	kg
Electronvolt	eV	$1.602\,176\,634 \times 10^{-19}$	$\mathrm{kg \cdot m^2 \cdot s^{-2}}$

Hydrogen and galaxies (10 points)

This problem aims to study the peculiar physics of galaxies, such as their dynamics and structure. In particular, we explain how to measure the mass distribution of our galaxy from the inside. For this we will focus on hydrogen, its main constituent.

Throughout this problem we will only use $\hbar$, defined as $\hbar = h/2\pi$.

Part A - Introduction

Bohr model

We assume that the hydrogen atom consists of a non-relativistic electron, with mass m_e, orbiting a fixed proton. Throughout this part, we assume its motion is on a circular orbit.

A.1	Determine the electron's velocity v in a circular orbit of radius r.	0.2pt

In the Bohr model, we assume the magnitude of the electron's angular momentum L is quantized, $L = n\hbar$ where $n > 0$ is an integer. We define $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx 7.27 \times 10^{-3}$.

A.2	Show that the radius of each orbit is given by $r_n = n^2 r_1$, where r_1 is called the Bohr radius. Express r_1 in terms of α, m_e, c and $\hbar$ and calculate its numerical value with 3 digits. Express v_1, the velocity on the orbit of radius r_1, in terms of α and c.	0.5pt

A.3	Determine the electron's mechanical energy E_n on an orbit of radius r_n in terms of e, ε_0, r_1 and n. Determine E_1 in the ground state in terms of α, m_e and c. Compute its numerical value in eV.	0.5pt

Hydrogen fine and hyperfine structures

The rare spontaneous inversion of the electron's spin causes a photon to be emitted on average once per 10 million years per hydrogen atom. This emission serves as a hydrogen tracer in the universe and is thus fundamental in astrophysics. We will study the transition responsible for this emission in two steps.

First, consider the interaction between the electron spin and the relative motion of the electron and the proton. Working in the electron's frame of reference, the proton orbits the electron at a distance r_1. This produces a magnetic field $\vec{B}_1$.

A.4	Determine the magnitude B_1 of $\vec{B}_1$ at the position of the electron in terms of μ_0, e, α, c and r_1.	0.5pt

Second, the electron spin creates a magnetic moment $\vec{\mathscr{M}}_s$. Its magnitude is roughly $\mathscr{M}_s = \frac{e}{m_e}\hbar$. The *fine* (F) structure is related to the energy difference ΔE_{F} between an electron with a magnetic moment $\vec{\mathscr{M}}_s$ parallel to $\vec{B}_1$ and that of an electron with $\vec{\mathscr{M}}_s$ anti-parallel to $\vec{B}_1$. Similarly, the *hyperfine* (HF) structure is related to the energy difference ΔE_{HF}, due to the interaction between parallel and anti-parallel magnetic moments of the electron and the proton. It is known to be approximately $\Delta E_{\mathrm{HF}} \simeq 3.72 \frac{m_e}{m_p} \Delta E_{\mathrm{F}}$ where m_p is the proton mass.

A.5	Express ΔE_{F} as a function of α and E_1. Express the wavelength λ_{HF} of a photon emitted during a transition between the two states of the hyperfine structure and give its numerical value with two digits.	0.5pt

Part B - Rotation curves of galaxies

Data

- Kiloparsec: $1\,\mathrm{kpc} = 3.09 \times 10^{19}\,\mathrm{m}$
- Solar mass : $1\,\mathrm{M}_{\odot} = 1.99 \times 10^{30}\,\mathrm{kg}$

We consider a spherical galaxy centered around a fixed point O. At any point P, let $\rho = \rho(P)$ be the volumetric mass density and $\varphi = \varphi(P)$ the associated gravitational potential (i.e. potential energy per unit mass). Both ρ and φ depend only on $r = \left\|\overrightarrow{OP}\right\|$. The motion of a mass m located at P, due to the field φ, is restricted to a plane containing O.

B.1	In the case of a circular orbit, determine the velocity v_c of an object on a circular orbit passing through P in terms of r and $\dfrac{d\varphi}{dr}$.	0.2pt

Fig. 1(A) is a picture of the spiral galaxy NGC 6946 in the visible band (from the $0.8\,\mathrm{m}$ Schulman Telescope at the Mount Lemmon Sky Center in Arizona). The little ellipses in Fig. 1(B) show experimental measurements of v_c for this galaxy. The central region ($r < 1\,\mathrm{kpc}$) is named the bulge. In this region, the mass distribution is roughly homogeneous. The red curve is a prediction for v_c if the system were homogeneous in the bulge and keplerian ($\varphi(r) = -\beta/r$ with $\beta > 0$) outside it, i.e. considering that the total mass of the galaxy is concentrated in the bulge.

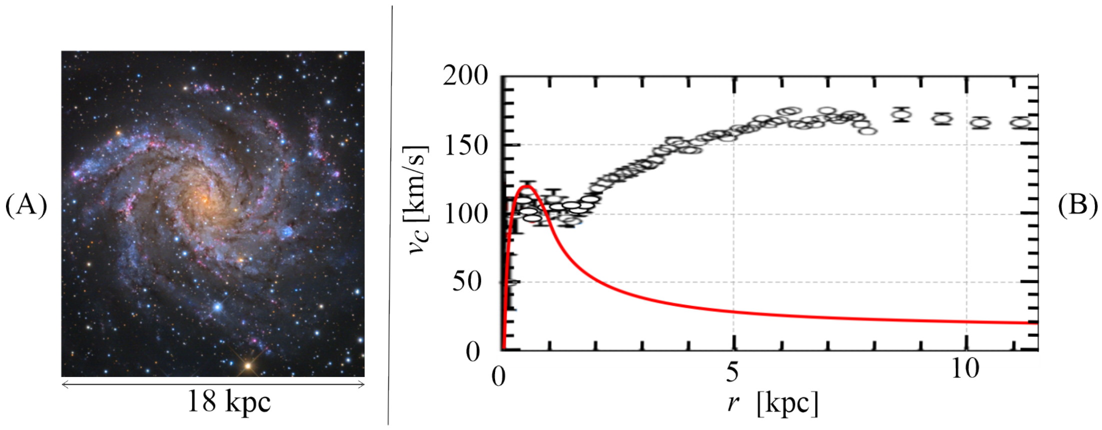


Fig. 1: NGC 6946 galaxy: Picture (A) and rotation curve (B).

B.2	Deduce the mass M_b of the bulge of NGC 6946 from the red rotation curve in Fig. 1(B), in solar mass units.	0.5pt

Comparing the keplerian model and the experimental data makes astronomers confident that part of the mass is invisible in the picture. They thus suppose that the galaxy's actual mass density is given by

$$\rho_m(r) = \frac{C_m}{r_m^2 + r^2} \tag{1}$$

where $C_m > 0$ and $r_m > 0$ are constants.

B.3	Show that the velocity profile $v_{c,m}(r)$, corresponding to the mass density in Eq. 1, can be written $v_{c,m}(r) = \sqrt{k_1 - \frac{k_2 \cdot \arctan(\frac{r}{r_m})}{r}}$. Express k_1 and k_2 in terms of C_m, r_m and G. (Hints: $\int_0^r \frac{x^2}{a^2 + x^2} dx = r - a \arctan(r/a)$, and: $\arctan(x) \simeq x - x^3/3$ for $x \ll 1$.) Simplify $v_{c,m}(r)$ when $r \ll r_m$ and when $r \gg r_m$. Show that if $r \gg r_m$, the mass $M_m(r)$ embedded in a sphere of radius r with the mass density given by Eq. 1 simplifies and depends only on C_m and r. Estimate the mass of the galaxy NGC 6946 actually present in the picture in Fig. 1(A).	1.8pt

Part C - Mass distribution in our galaxy

For a spiral galaxy, the model for Eq. 1 is modified and one usually considers the gravitational potential is given by $\varphi_G(r,z) = \varphi_0 \ln\left(\frac{r}{r_0}\right) \exp\left[-\left(\frac{z}{z_0}\right)^2\right]$, where z is the distance to the galactic plane (defined by $z = 0$), and $r < r_0$ is now the axial radius and $\varphi_0 > 0$ a constant to be determined. r_0 and z_0 are constant values.

C.1	Find the equation of motion on z for the vertical motion of a point mass m in such a potential, assuming r is constant. Show that, if $r < r_0$, the galactic plane is a stable equilibrium state by giving the angular frequency ω_0 of small oscillations around it.	0.5pt

From here on, we set $z = 0$.

C.2	Identify the regime, either $r \gg r_m$or $r \ll r_m$, in which the model of Eq. 1 recovers a potential of the form $\varphi_G(r,0)$ with a suitable definition of φ_0. Under this condition $v_c(r)$ no longer depends on r. Express it in terms of φ_0.	0.6pt

Therefore, outside the bulge the velocity modulus v_c does not depend on the distance to the galactic center. We will use this fact, as astronomers do, to measure the galaxy's mass distribution from the inside.

All galactic objects considered here for astronomical observations, such as stars or nebulae, are primarily composed of hydrogen. Outside the bulge, we assume that they rotate on circular orbits around the galactic center C. S is the sun's position and E that of a given galactic object emitting in the hydrogen spectrum. In the galactic plane, we consider a line of sight SE corresponding to the orientation of an observation, on the unit vector $\hat{u}_v$ (see Fig. 2).

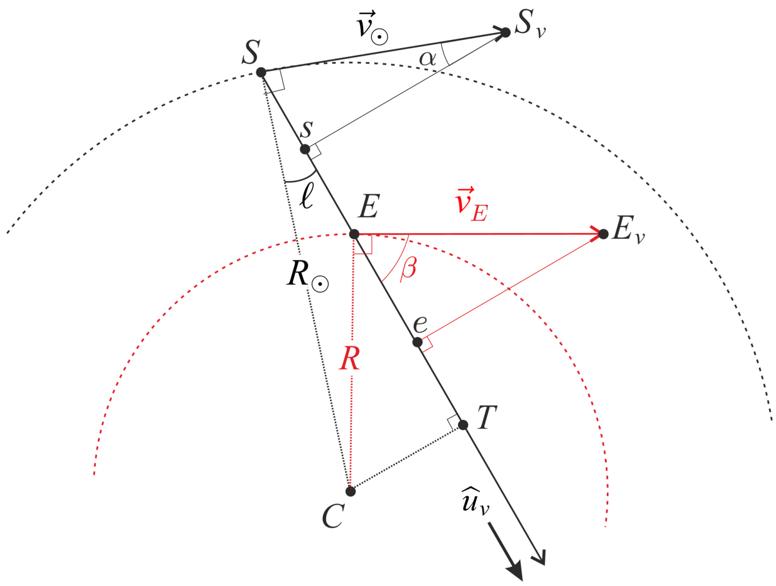


Fig. 2: Geometry of the measurement

Let ℓ be the galactic longitude, measuring the angle between SC and the SE. The sun's velocity on its circular orbit of radius $R_\odot = 8.00\,\mathrm{kpc}$ is denoted $\vec{v}_\odot$. A galactic object in E orbits on another circle of radius R at velocity $\vec{v}_E$. Using a Doppler effect on the previously studied $21\,\mathrm{cm}$ line, one can obtain the relative radial velocity $v_{rE/S}$ of the emitter E with respect to the sun S : it is the projection of $\vec{v}_E - \vec{v}_\odot$ on the line of sight.

C.3	Determine $v_{rE/S}$ in terms of ℓ, R, $R_\odot$ and $v_\odot$. Then, express R in terms of $R_\odot$, $v_\odot$, ℓ and $v_{rE/S}$.	0.7pt

Using a radio telescope, we make observations in the plane of our galaxy toward a longitude $\ell = 30°$. The frequency band used contains the $21\,\mathrm{cm}$ line, whose frequency is $f_0 = 1.42\,\mathrm{GHz}$. The results are reported in Fig. 3.

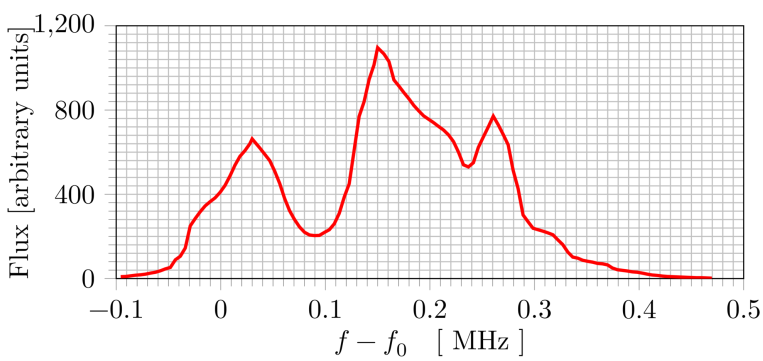


Fig. 3: Electromagnetic signal as a function of the frequency shift, measured in the radio frequency band at $\ell = 30°$ using EU-HOU RadioAstronomy

C.4	In our galaxy, $v_{\odot} = 220\,\mathrm{km \cdot s^{-1}}$. Determine the values of the relative radial velocity (with 3 significant digits) and the distance from the galactic center (with 2 significant digits) of the 3 sources observed in Fig. 3. Distances should be expressed as multiples of $R_{\odot}$.	0.6pt

C.5	On the top view of our galaxy (in the answer box), indicate the positions of the sources observed in Fig. 3. What could be deduced from repeated measurements changing ℓ?	0.6pt

Part D - Tully-Fisher relation and MOND theory

The flat external velocity curve of NGC 6946 in Fig. 1 is a common property of spiral galaxies, as can be seen in Fig. 4 (left). Plotting the external constant velocity value $v_{c,\infty}$ as a function of the measured total mass M_{tot} of each galaxy gives an interesting correlation called the Tully-Fischer relation, see Fig. 4 (right).

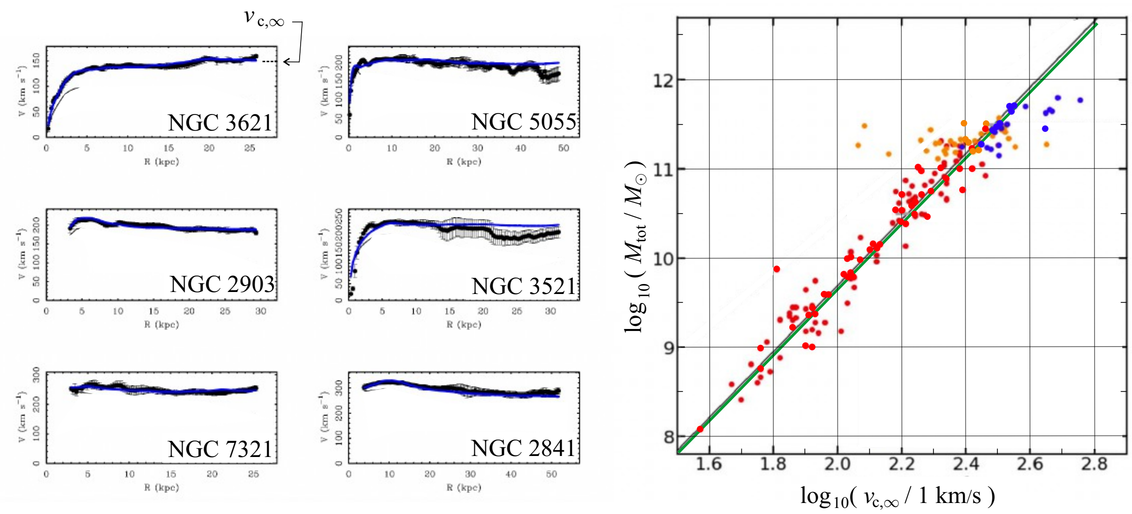


Fig. 4. Left: Rotation curves for typical spiral galaxies - Right: $\log_{10}(M_{\text{tot}})$ as a function of $\log_{10}(v_{c,\infty})$ on linear scales. Colored dots correspond to different galaxies and different surveys. The green line is the Tully-Fischer relation which is in very good agreement with the best fit line of the data (in black).

D.1	Assuming that the radius R of a galaxy doesn't depend on its mass, show that the model of Eq. 1 (part B) gives a relation of the form $M_{\text{tot}} = \eta v_{c,\infty}^{\gamma}$ where γ and η should be specified. Compare this expression to the Tully-Fischer relation by computing γ_{TF}.	0.4pt

In the extremely low acceleration regime, of the order of $a_0 = 10^{-10}\,\mathrm{m \cdot s^{-2}}$, the MOdified Newtonian Dynamics (MOND) theory suggests that one can modify Newton's second law using $\vec{F} = m\mu\left(\frac{a}{a_0}\right)\vec{a}$ where $a = \|\vec{a}\|$ is the modulus of the acceleration and the μ function is defined by $\mu(x) = \frac{x}{1+x}$.

D.2	Using data for NGC 6946 in Fig. 1, estimate, within Newton's theory, the modulus of the acceleration a_m of a mass in the outer regions of NGC 6946.	0.2pt

D.3	Let m be a mass on a circular orbit of radius r with velocity $v_{c,\infty}$ in the gravity field of a fixed mass M. Within the MOND theory, with $a \ll a_0$, determine the Tully-Fischer exponent. Using data for NGC 6946 and/or Tully-Fischer law, calculate a_0 to show that MOND operates in the correct regime.	0.8pt

D.4	Considering relevant cases, determine $v_c(r)$ for all values of r in the MOND theory in the case of a gravitational field due to a homogeneously distributed mass M with radius R_b.	0.9pt

Cox's Timepiece (10 points)

In 1765, British clockmaker James Cox invented a clock whose only source of energy is the fluctuations in atmospheric pressure. Cox's clock used two vessels containing mercury. Changes in atmospheric pressure caused mercury to move between the vessels, and the two vessels to move relative to each other. This movement acted as an energy source for the actual clock.

We propose an analysis of this device. Throughout, we assume that

- the Earth's gravitational field $\vec{g} = -g\,\vec{u_z}$ is uniform with $g = 9.8\,\mathrm{m \cdot s^{-2}}$ and $\vec{u_z}$ a unit vector;
- all liquids are incompressible and their density is denoted ρ;
- no surface tension effects will be considered;
- the variations of atmospheric pressure with altitude are neglected;
- the surrounding temperature T_a is uniform and all transformations are isothermal.

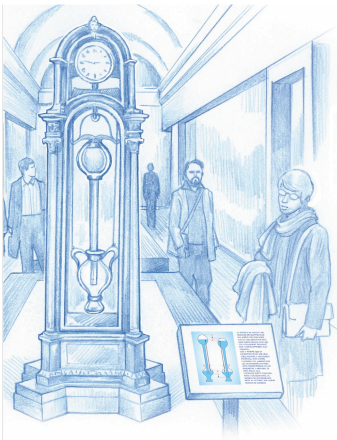


Fig. 1. Artistic view of Cox's clock [1]

Part A - Pulling on a submerged tube

We first consider a bath of water that occupies the semi-infinite space $z \le 0$. The air above it is at a pressure $P_a = P_0$. A cylindrical vertical tube of length $H = 1\,\mathrm{m}$, cross-sectional area $S = 10\,\mathrm{cm}^2$ and mass $m = 0.5\,\mathrm{kg}$ is dipped into the bath. The bottom end of the tube is open, and the top end of the tube is closed. We denote h the altitude of the top of the tube and z_ℓ that of the water inside the tube. The thickness of the tube walls is neglected.

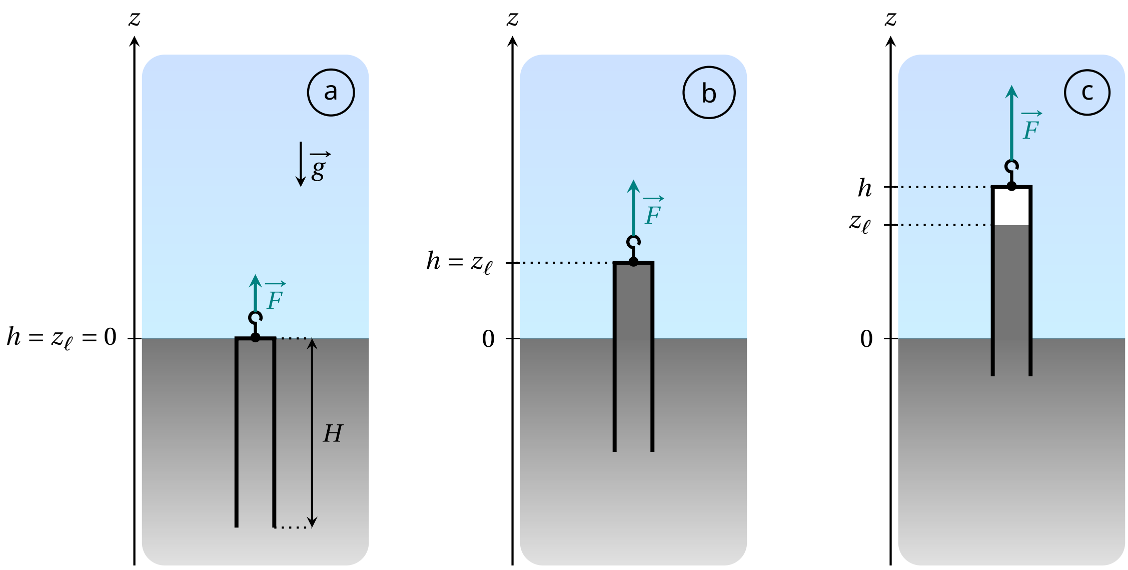


Fig. 2. Sketch of the tube in different configurations

We start from the situation where the tube in Fig. 2 contains no gas and its top is at the bath level: in other words, $h = 0$ and $z_\ell = 0$ (case a). The tube is then slowly lifted until its bottom end reaches the bath level. The pulling force exerted on the tube is denoted $\vec{F} = F\,\vec{u_z}$.

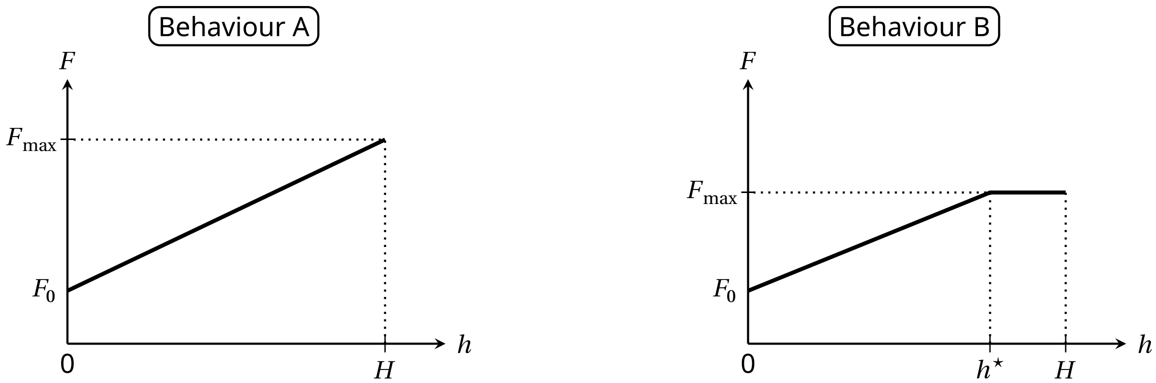
A.1	For the configuration shown in Fig. 2 (case b), express the pressure P_w in the water at the top of the tube. Also express the force $\vec{F}$ necessary to maintain the tube at this position. Expressions must be written in terms of P_0, ρ, m, S, h, g and $\vec{u_z}$.	0.2pt

Three experiments are performed. In each, the tube is lifted from the initial state shown in Fig. 2(a) under the conditions specified in Table 1.

Experiment	Liquid	T_a (°C)	ρ (kg·m^{-3})	P_{sat} (Pa)
1	Water	20	1.00×10^3	2.34×10^3
2	Water	80	0.97×10^3	47.4×10^3
3	Water	99	0.96×10^3	99.8×10^3

Table 1. Experimental conditions and numerical values of physical quantities for each experiment (P_{sat} designates the saturated vapour pressure of the pure fluid)

In each case, we study the evolution of the force F that must be applied in order to maintain the tube in equilibrium at an altitude h, the external pressure being fixed at $P_a = P_0 = 1.000 \times 10^5\,\mathrm{Pa}$. Two different behaviours are possible



A.2	For each experiment, complete the table in the answer sheet to indicate the expected behaviour and the numerical values for F_{max} and for $h^\star$ (when pertinent), where F_{max} and $h^\star$ are defined in the figures illustrating the two behaviours.	0.8pt

When we replace the water with liquid mercury (whose properties are given below), behaviour B is observed.

Liquid	T_a (°C)	ρ (kg·m^{-3})	P_{sat} (Pa)
Mercury	20	13.5×10^3	0.163

A.3	Express the relative error, denoted ε, committed when we evaluate the maximal force $F_{\max}$ neglecting P_{sat} compared to P_0. Give the numerical value of ε.	0.3pt

Part B - Two-part barometric tube

From now on, we work with mercury (density $\rho = 13.5 \times 10^3\,\text{kg}\cdot\text{m}^{-3}$) at the ambient temperature $T_{\text{a}} = 20\,°\text{C}$ and we take $P_{\text{sat}} = 0$.

Let us consider a tube with a reservoir on top, modeled as two superposed cylinders of different dimensions, as shown in Fig. 3.

- the bottom part (still called the tube) has cross-sectional area S_{t} and height $H_{\text{t}} = 80\,\text{cm}$;
- the top part (called the bulb) has cross-sectional area $S_{\text{b}} > S_{\text{t}}$ and height $H_{\text{b}} = 20\,\text{cm}$.

This two-part tube is dipped into a semi-infinite liquid bath.

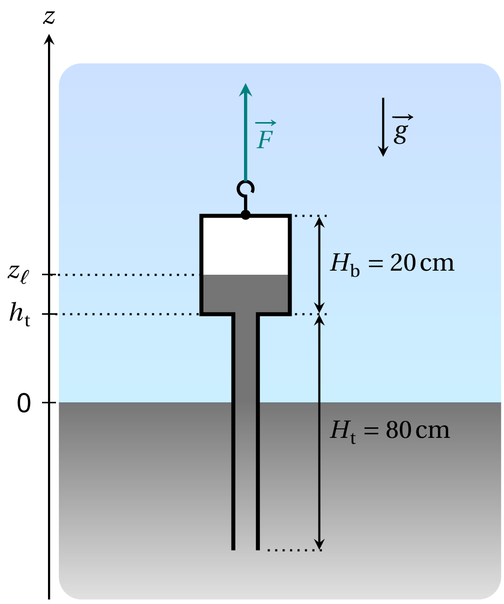


Fig. 3. Sketch of the two-part barometric tube

As in Part A, the system is prepared such that the tube contains no air. We identify the vertical position of the tube by the altitude h_{t} of the junction between the tube and the bulb. The height of the column of mercury is again denoted z_ℓ. The force $\vec{F}$ that must be exerted to maintain the tube in equilibrium in the configuration shown in Fig. 3 can now be written as

$$\vec{F} = (m_{\text{tb}} + m_{\text{add}})\, g\, \vec{u_z} \tag{1}$$

where m_{tb} is the total mass of the two-part tube (when empty of mercury).

B.1	On the answer sheet, color the area corresponding to the volume of liquid mercury that is responsible for the term m_{add} appearing in equation (1).	0.3pt

The mass m_{add} depends both on the height h_{t} and the atmospheric pressure P_{a}. For the next question, assume that the atmospheric pressure is fixed at $P_{\text{a}} = P_0 = 1.000 \times 10^5\,\text{Pa}$. Starting from the situation where the system is completely submerged, the tube is slowly lifted until its base is flush with the liquid bath.

B.2	Sketch the evolution of the mass m_{add} as a function of h_{t} for $h_{\text{t}} \in [-H_{\text{b}}, H_{\text{t}}]$. On the graph, provide the expression for the slopes of the different segments, as well as the h_{t} analytical value of any angular points, in terms of P_0, ρ, g, S_{b}, S_{t}, H_{b} and H_{t}.	1.4pt

As the system is lifted while $P_a = P_0 = 10^5\,\mathrm{Pa}$, we stop when the free surface of the liquid is in the middle of the bulb. The value of h_t is fixed and then we observe variations in the mass m_{add} due to variations in the atmospheric pressure described by

$$P_a(t) = P_0 + P_1(t) \tag{2}$$

where P_0 designates the average value and P_1 is a perturbative term. We model P_1 by a periodic triangular function of amplitude $A = 5 \times 10^2\,\mathrm{Pa}$ and period τ_1 of 1 week.

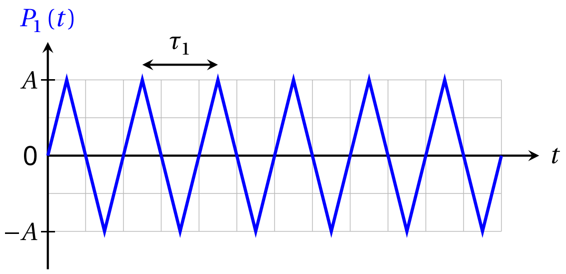


Fig. 4. Simplified model of the perturbative term $P_1(t)$

B.3	Given that $S_t = 5\,\mathrm{cm}^2$ and $S_b = 200\,\mathrm{cm}^2$, express the amplitude Δm_{add} of the variations of the mass m_{add} over time, then give its numerical value. *Assume that the liquid surface always stays in the bulb.*	0.3pt

Part C - Cox's timepiece

The real mechanism developed by Cox is complex (Fig. 5). We study a simplified version, depicted in Fig. 6, and described below

- a cylindrical bottom cistern containing a mercury bath ;
- a two-part barometric tube identical to that studied in part B, which is still completely emptied of any air, is dipped into the bath ;
- the cistern and the two-part tube are each suspended by a cable. Both cables (assumed to be inextensible and of negligible mass) pass through a system of ideal pullies and finish attached to either side of the same mass M, which can slide on a horizontal surface ;
- the total volume of liquid mercury contained in the system is $V_\ell = 5\,\mathrm{L}$.

The height, cross-section and masses of each part are given in Table 2. The position of mass M is referenced by the coordinate x of its center of mass. We consider solid friction between the horizontal support and the mass M, without distinction between static and dynamic coefficients; the magnitude of this force when sliding occurs is denoted F_s.

Two stops limit the displacement of the mass M such that $-X \le x \le X$ (with $X > 0$). Assume that the value of X guarantees that

- the bottom of the two-part tube never touches the bottom of the cistern nor comes out of the liquid bath;
- the altitude z_ℓ of the mercury column is always in the upper bulb.

Fig. 5. Real Cox's timepiece [2] (without mercury)

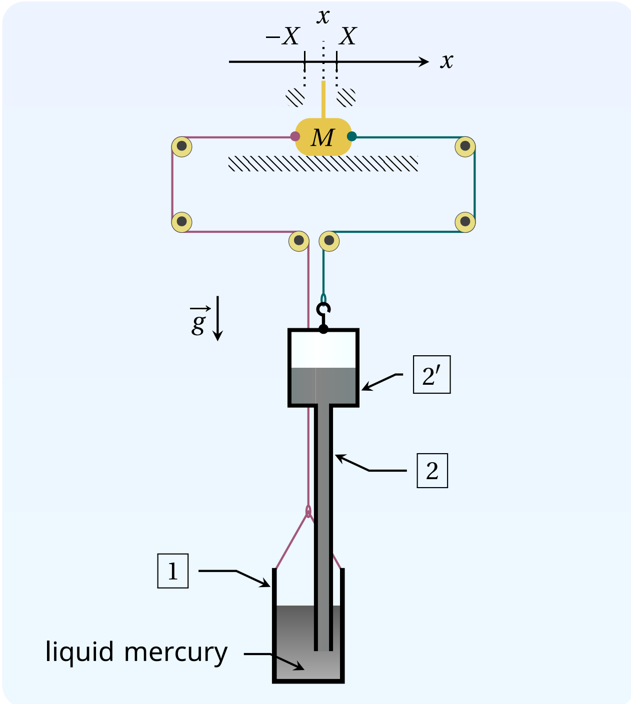


Fig. 6. Sketch of the system modeling the timepiece

Reference	Name	Height	Cross section area	Empty mass
1	cistern	$H_c = 30\,\mathrm{cm}$	$S_c = 210\,\mathrm{cm}^2$	m_c
2	tubular part of the barometric tube	$H_t = 80\,\mathrm{cm}$	$S_t = 5\,\mathrm{cm}^2$	total mass of the barometric tube : m_{tb}
2′	bulb of the barometric tube	$H_b = 20\,\mathrm{cm}$	$S_b = 200\,\mathrm{cm}^2$	

Table 2. Dimensions and notations for the model system

The system evolves in contact with the atmosphere, whose pressure fluctuates as in Fig. 4 (still with amplitude $A = 5 \times 10^2\,\mathrm{Pa}$ and period $\tau_1 = 1\,\mathrm{week}$). At the start $t = 0$, the mass M is at rest at $x = 0$ and the tensions exerted by the two cables on either side of the mass M are in balance while $P_1(0) = 0$. We define

$$\xi = \frac{S_b + S_c - S_t}{S_b S_c}\frac{F_s}{A} \simeq \frac{S_b + S_c}{S_b S_c}\frac{F_s}{A} \tag{3}$$

where the last expression uses that $S_t \ll S_b, S_c$ (which we will assume is valid until the end of the problem).

C.1	Determine the threshold $\xi^\star$ such that M remains indefinitely at rest when $\xi > \xi^\star$.	1pt

For the next question only, suppose that the mass M is temporarily blocked at $x = X$.

C.2	Give an expression for the total tension force $\vec{T} = T\,\vec{u_x}$ acting on the mass M due to the tension in two cables at this position, when $P_1 = 0$, in terms of ρ, g, X and pertinent cross-sections.	1pt

When $\xi < \xi^\star$, starting again from $x = 0$ and $P_1 = 0$, two different behaviours can be observed for $t \geq 0$. To distinguish them, we need to introduce another parameter

$$\lambda = \frac{2\left(S_\mathrm{b} - S_\mathrm{t}\right)}{S_\mathrm{b}}\,\frac{\rho\, g\, X}{A} \simeq \frac{2\,\rho\, g\, X}{A} \tag{4}$$

C.3	Complete the table in the answer sheet to indicate the condition under which each regime is obtained. Conditions must be expressed as inequalities on ξ and/or λ. In addition, sketch the variations of $x\,(t)\,/X$ for $t \in \left[0,\, 3\,\tau_1\right]$ that are consistent with the variations of $P_1\,(t)\,/A$ already present. *Specification of remarkable points coordinates is not required.*	2pt

In the real Cox's timepiece, energy provided by the mechanism is stored using a system of ratchets and used to raise a counterweight, like in a traditional clock. In the simplified model studied here, the energy recovered by the clock corresponds to the energy dissipated by the friction force exerted by the horizontal surface on the mass M. From now on, we assume that the system is dimensioned such that to work in the regime that allows the clock to recuperate energy. We also assume that the permanent regime is established. We denote W the energy dissipated by the solid friction force during a period τ_1, which can be expressed only in terms of F_s and X.

All else equal, F_s and X can be adjusted to maximize the energy W; we denote $F_\mathrm{s}^\star$ and $X^\star$ their respective values in the optimal situation.

C.4	Considering $S_\mathrm{b} \simeq S_\mathrm{c}$ and $S_\mathrm{t} \ll S_\mathrm{b}$, determine the expressions for $F_\mathrm{s}^\star$ and $X^\star$ as functions of ρ, g, S_c and A. Express the corresponding maximum energy $W^\star$, then calculate its numerical value with $A = 5 \times 10^2\,\mathrm{Pa}$.	1pt

We denote $W_\mathrm{pr}^\star$ the work of atmospheric pressure forces received by the system in the optimal situation during a period τ_1.

C.5	Express $W_\mathrm{pr}^\star$, then calculate the ratio $W^\star / W_\mathrm{pr}^\star$. *It could be useful to represent the evolution of the system in a (P, V) diagram, where V is the system's volume.*	1.7pt

Credits:

[1]: Bruno Vacaro;

[2]: Victoria and Albert Museum, London.

Champagne! (10 points)

Warning: Excessive alcohol consumption is harmful to health and drinking alcohol below legal age is prohibited.

Champagne is a French sparkling wine. Fermentation of sugars produces carbon dioxide ($\mathrm{CO_2}$) in the bottle. The molar concentration of $\mathrm{CO_2}$ in the liquid phase c_ℓ and the partial pressure $P_{\mathrm{CO_2}}$ in the gas phase are related by $c_\ell = k_\mathrm{H} P_{\mathrm{CO_2}}$, known as Henry's law and where k_H is called Henry's constant.

Fig. 1. A glass filled with champagne.

Data

- Surface tension of champagne $\sigma = 47 \times 10^{-3}\,\mathrm{J \cdot m^{-2}}$
- Density of the liquid $\rho_\ell = 1.0 \times 10^{3}\,\mathrm{kg \cdot m^{-3}}$
- Henry's constant at $T_0 = 20\,°\mathrm{C}$, $k_\mathrm{H}(20\,°\mathrm{C}) = 3.3 \times 10^{-4}\,\mathrm{mol \cdot m^{-3} \cdot Pa^{-1}}$
- Henry's constant at $T_0 = 6\,°\mathrm{C}$, $k_\mathrm{H}(6\,°\mathrm{C}) = 5.4 \times 10^{-4}\,\mathrm{mol \cdot m^{-3} \cdot Pa^{-1}}$
- Atmospheric pressure $P_0 = 1\,\mathrm{bar} = 1.0 \times 10^{5}\,\mathrm{Pa}$
- Gases are ideal with an adiabatic coefficient $\gamma = 1.3$

Part A. Nucleation, growth and rise of bubbles

Immediately after opening a bottle of champagne at temperature $T_0 = 20\,°\mathrm{C}$, we fill a glass. The pressure in the liquid is P_0 and its temperature stays constant at T_0. The concentration c_ℓ of dissolved $\mathrm{CO_2}$ exceeds the equilibrium concentration and we study the nucleation of a $\mathrm{CO_2}$ bubble. We note a its radius and P_b its inner pressure.

A.1	Express the pressure P_b in terms of P_0, a and σ.	0.2pt

In the liquid, the concentration of dissolved $\mathrm{CO_2}$ depends on the distance to the bubble. At long distance we recover the value c_ℓ and we note c_b the concentration close to the bubble surface. According to Henry's law, $c_\mathrm{b} = k_\mathrm{H} P_\mathrm{b}$. We furthermore assume in all the problem that bubbles contain only $\mathrm{CO_2}$.

Since $c_\ell \neq c_\mathrm{b}$, $\mathrm{CO_2}$ molecules diffuse from areas of high to low concentration. We assume also that any molecule from the liquid phase reaching the bubble surface is transferred to the vapour.

A.2	Express the critical radius a_c above which a bubble is expected to grow in terms of P_0, σ, c_ℓ and c_0 where $c_0 = k_\mathrm{H} P_0$. Calculate numerically a_c for $c_\ell = 4c_0$.	0.5pt

In practice, bubbles mainly grow from pre-existing gas cavities. Consider then a bubble with initial radius $a_0 \approx 40\,\mathrm{\mu m}$. The number of moles of $\mathrm{CO_2}$ transferred at the bubble's surface per unit area and time is noted j. Two models are possible for j.

- model (1) $j = \dfrac{D}{a}(c_\ell - c_\mathrm{b})$ where D is the diffusion coefficient of $\mathrm{CO_2}$ in the liquid.
- model (2) $j = K(c_\ell - c_\mathrm{b})$ where K is a constant here.

Experimentally, the bubble radius $a(t)$ is found to depend on time as shown in **Fig. 2**. Here $c_\ell \approx 4c_0$, and since bubbles are large enough to be visible, the excess pressure due to surface tension can be neglected and $P_\mathrm{b} \approx P_0$.

A.3	Express the number of CO_2 moles in the bubble n_c in terms of a, P_0, T_0 and ideal gas constant R. Find $a(t)$ for both models. Indicate which model explains the experimental results in **Fig. 2**. Depending on your answer, calculate numerically K or D.	1.2pt

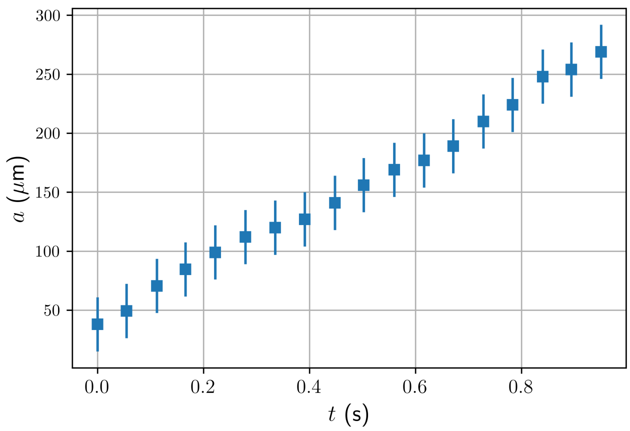


Fig. 2. Time evolution of CO_2 bubble radius in a glass of champagne (*adapted from [1]*).

Eventually bubbles detach from the bottom of the glass and continue to grow while rising. **Fig. 3.** shows a train of bubbles. The bubbles of the train have the same initial radius and are emitted at a constant frequency $f_b = 20\,\mathrm{Hz}$.

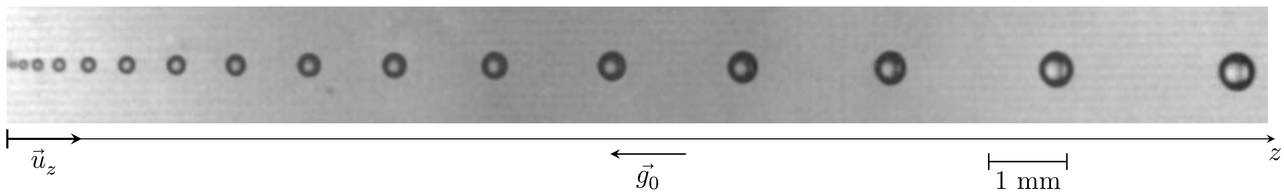


Fig. 3. A train of bubbles. The photo is rotated horizontally for the page layout *(adapted from [1])*.

For the range of velocities studied here, the drag force F on a bubble of radius a moving at velocity v in a liquid of dynamic viscosity η is given by Stokes' law $F = 6\pi\eta a v$. Measurements show that at any moment in time, the bubble can be assumed to be travelling at its terminal velocity.

A.4	Give the expression of the main forces exerted on a vertically rising bubble. Obtain the expression of $v(a)$. Give a numerical estimate of η using ρ_ℓ, g_0 and quantities measured on **Fig. 3.**	0.8pt

The quasi-stationary growth of bubbles with rate $q_a = \frac{\mathrm{d}a}{\mathrm{d}t}$ still applies during bubble rise.

A.5	Express the radius a_{H_ℓ} of a bubble reaching the free surface in terms of height travelled H_ℓ, growth rate $q_a = \frac{\mathrm{d}a}{\mathrm{d}t}$, and any constants you may need. Assume $a_{H_\ell} \gg a_0$ and q_a constant, and give the numerical value of a_{H_ℓ} with $H_\ell = 10\,\mathrm{cm}$ and q_a corresponding to **Fig. 2**.	0.5pt

There are N_{b} nucleation sites of bubbles. Assume that the bubbles are nucleated at a constant frequency f_{b} at the bottom of a glass of champagne (height H_ℓ for a volume V_ℓ), with a_0 still negligible. Neglect diffusion of $\mathrm{CO_2}$ at the free surface.

A.6	Write the differential equation for $c_\ell(t)$. Obtain from this equation the characteristic time τ for the decay of the concentration of dissolved $\mathrm{CO_2}$ in the liquid.	1.1pt

Part B. Acoustic emission of a bursting bubble

Small bubbles are nearly spherical as they reach the free surface. Once the liquid film separating the bubble from the air thins out sufficiently, a circular hole of radius r forms in the film and, driven by surface tension, opens very quickly (**Fig. 4.** left). The hole opens at constant speed v_{f} (**Fig. 4.** right). The film outside the rim remains still, with constant thickness h.

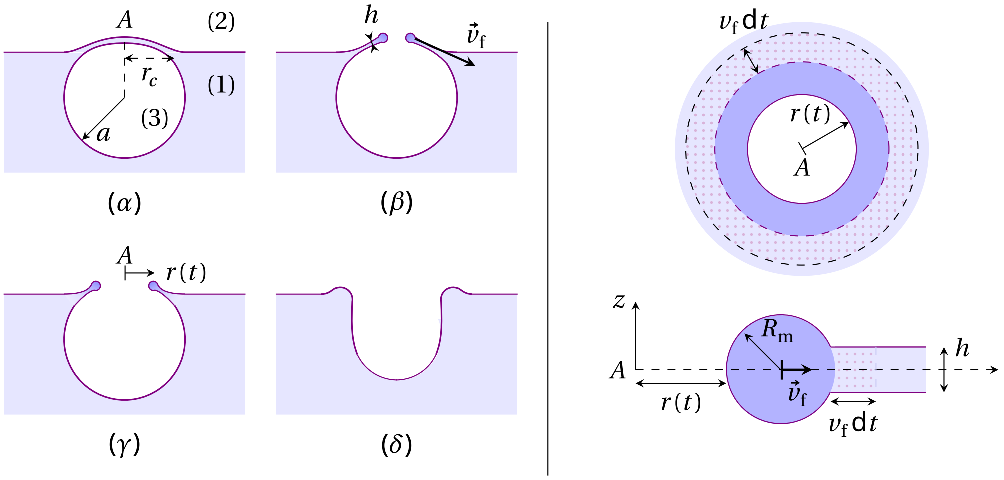


Fig. 4. *(Left)* (α) Bubble at the surface: (1) liquid, (2) air at pressure P_0 and (3), $\mathrm{CO_2}$ at pressure P_{b}, (β) and (γ) retraction of the liquid film, where the rim is in dark blue, (δ) bubble collapse. *(Right)* Retraction of the liquid film at time t. Top: sketch of the pierced film seen from above. Bottom: cross-section of the rim and the retracting film. During $\mathrm{d}t$ the rim accumulates nearby liquid (dotted).

Due to dissipative processes, only half of the difference of the surface energy between t and $t+\mathrm{d}t$ of the rim and the accumulated liquid is transformed into kinetic energy. We further assume that the variation of the surface of the rim is negligible compared to that of the film.

B.1	Express v_{f} in terms of ρ_ℓ, σ and h.	1.1pt

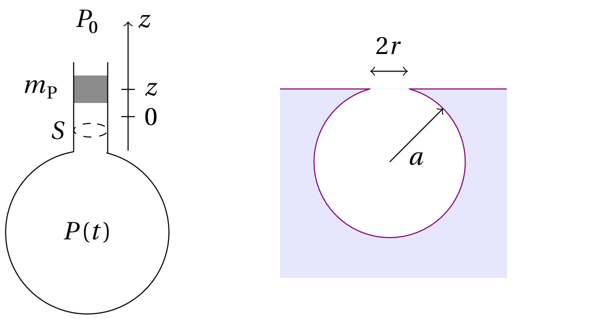


Fig. 5. *(Left)* a Helmholtz resonator. *(Right)* a bubble as an oscillator.

When the film bursts, it releases internal pressure and emits a sound. We model this acoustic emission by a Helmholtz resonator: a cavity open to the atmosphere at P_0 through a bottleneck aperture of area S (**Fig. 5**. left). In the neck, a mass m_p makes small amplitude position oscillations due to the pressure forces it experiences as the gas in the cavity expands or compresses adiabatically. The gravity force on m_p is negligible compared to pressure forces. Let V_0 be the volume of gas under the mass m_p for $P = P_0$ as $z = 0$.

B.2	Express the frequency of oscillation f_0 of m_p. Hint: for $\varepsilon \ll 1$, $(1+\varepsilon)^\alpha \approx 1 + \alpha\varepsilon$.	1.1pt

The Helmholtz model may be used for a bubble of radius a. V_0 is the volume of the closed bubble. From litterature, the mass of the equivalent of the piston is $m_p = 8\rho_g r^3/3$ where r is the radius of the circular aperture and $\rho_g = 1.8\,\mathrm{kg\cdot m^{-3}}$ is the density of the gas (**Fig. 5**. right). During the bursting process, r goes from 0 to r_c, given by $r_\mathrm{c} = \dfrac{2}{\sqrt{3}} a^2 \sqrt{\dfrac{\rho_\ell g_0}{\sigma}}$. At the same time, the frequency of emitted sound increases until a maximum value of $40\,\mathrm{kHz}$ and the bursting time is $t_b = 3 \times 10^{-2}\,\mathrm{ms}$.

B.3	Find the radius a and the thickness h of the champagne film separating the bubble from the atmosphere.	1.1pt

Part C. Popping champagne

In a bottle, the total quantity of CO_2 is $n_\mathrm{T} = 0.2\,\mathrm{mol}$, either dissolved in the volume $V_\mathrm{L} = 750\,\mathrm{mL}$ of liquid champagne, or as a gas in the volume $V_\mathrm{G} = 25\,\mathrm{mL}$ under the cork (**Fig. 6**. left). V_G contains only CO_2. The equilibrium between both CO_2 phases follows Henry's Law. We suppose that the fast gaseous CO_2 expansion when the bottle is opened, is adiabatic and reversible. Ambient temperature T_0 and pressure $P_0 = 1\,\mathrm{bar}$ are constant.

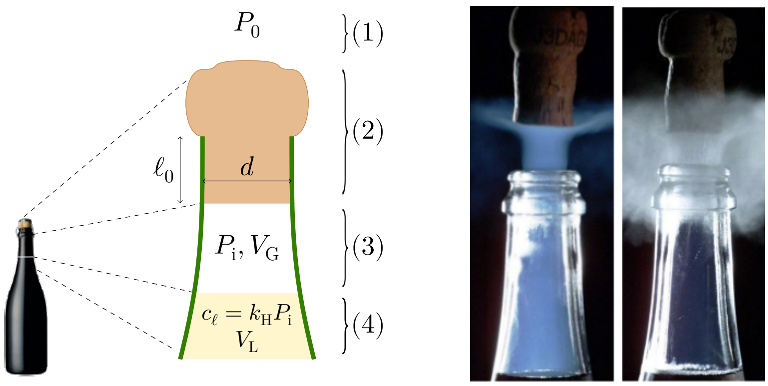


Fig. 6. *Left:* traditional bottleneck: (1) surrounding air, (2) cork stopper, (3) headspace, (4) liquid champagne. *Right*: Two phenomena observed while opening the bottle at two different temperatures (adapted from *[2]*).

C.1	Give the numerical value of the pressure P_i of gaseous $\mathrm{CO_2}$ in the bottle for $T_0 = 6\,°\mathrm{C}$ and $T_0 = 20\,°\mathrm{C}$.	0.4pt

Another step of champagne production (not described here) leads to the following values of P_i that we will use for the next questions: $P_\mathrm{i} = 4.69\,\mathrm{bar}$ at $T_0 = 6\,°\mathrm{C}$ and $P_\mathrm{i} = 7.45\,\mathrm{bar}$ at $T_0 = 20\,°\mathrm{C}$.

During bottle opening, two different phenomena can be observed, depending on T_0 (Fig. 6. right).

- either a blue fog appears, due to the formation of solid $\mathrm{CO_2}$ crystals (but water condensation is inhibited);
- or a grey-white fog appears, due to water vapor condensation in the air surrounding the bottleneck. In this latter case, there is no formation of $\mathrm{CO_2}$ solid crystals.

The saturated vapor pressure $P_\mathrm{sat}^{\mathrm{CO_2}}$ for the $\mathrm{CO_2}$ solid/gas transition follows : $\log_{10}\left(\frac{P_\mathrm{sat}^{\mathrm{CO_2}}}{P_0}\right) = A - \frac{B}{T+C}$ with T in K, $A = 6.81$, $B = 1.30 \times 10^3\,\mathrm{K}$ and $C = -3.49\,\mathrm{K}$.

C.2	Give the numerical value T_f of the $\mathrm{CO_2}$ gas at the end of the expansion, after opening a bottle, if $T_0 = 6\,°\mathrm{C}$ and if $T_0 = 20\,°\mathrm{C}$, if no phase transition occured. Choose which statements are true (several statements possible): 1. At $T_0 = 6\,°\mathrm{C}$ a grey-white fog appears while opening the bottle. 2. At $T_0 = 6\,°\mathrm{C}$ a blue fog appears while opening the bottle. 3. At $T_0 = 20\,°\mathrm{C}$ a grey-white fog appears while opening the bottle. 4. At $T_0 = 20\,°\mathrm{C}$ a blue fog appears while opening the bottle.	0.7pt

During bottle opening, the cork stopper pops out. We now determine the maximum height H_c it reaches. Assume that the friction force F due to the bottleneck on the cork stopper is $F = \alpha A$ where A is the area of contact and α is a constant to determine. Initially, the pressure force slightly overcomes the friction force. The cork's mass is $m = 10\,\mathrm{g}$, its diameter $d = 1.8\,\mathrm{cm}$ and the length of the cylindrical part initially stuck in the bottleneck is $\ell_0 = 2.5\,\mathrm{cm}$. Once the cork has left the bottleneck, you can neglect the net pressure force.

C.3	Give the numerical value of H_c if the external temperature is $T_0 = 6\,°\mathrm{C}$.	1.3pt

[1] Liger-Belair *et al,* Am. J. Enol. Vitic., Vol. 50, No. 3 (1999).

[2] Liger-Belair *et al.*, *Sc. Reports* **7**, 10938 (2017).