Part 1. Calibration

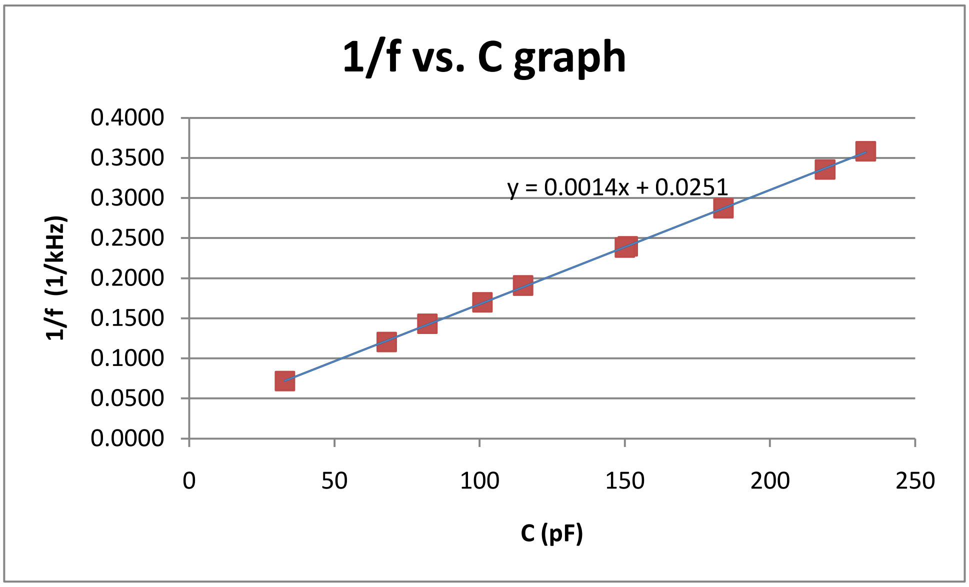
From the relationship between f and C given,

$$f = \frac{\alpha}{C + C_S} \qquad \Leftrightarrow \qquad \frac{1}{f} = \frac{1}{\alpha}C + \frac{C_S}{\alpha}$$

That is, theoretically, the graph of $\frac{1}{f}$ on the Y-axis versus C on the X-axis should be linear of which the slope and the Y-intercept is $\frac{1}{\alpha}$ and $\frac{C_S}{\alpha}$ respectively.

The table below shows the measured values of C (plotted on the X-axis,) f and, additionally, $\frac{1}{f}$, which is plotted on the Y-axis.

C (pF)	f (kHz)	1/f (ms)
33	13.94	0.0717
68	8.30	0.1205
82	6.99	0.1431
151	4.17	0.2398
233	2.79	0.3584
219	2.98	0.3356
184	3.48	0.2874
150	4.20	0.2381
115	5.24	0.1908
101	5.89	0.1698



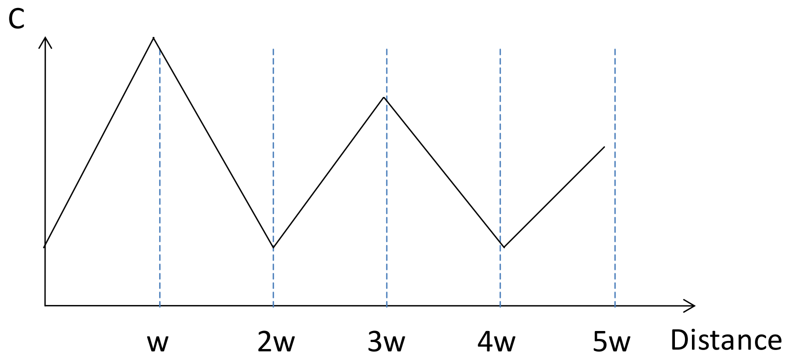
From this graph, the slope ($\frac{1}{\alpha}$) and the Y-intercept ($\frac{C_S}{\alpha}$) is equal to 0.0014 s/nF and 0.0251 ms respectively.

Hence, $\alpha = \frac{1}{\text{slope}} = \frac{1}{0.0014 \text{ s / nF}} = 714$ nF/s

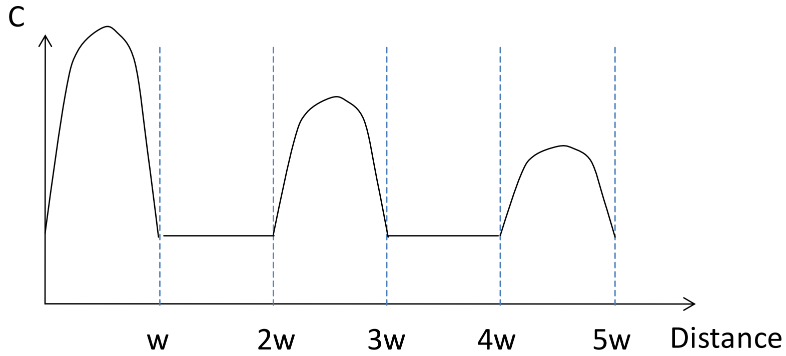
and $C_S = \frac{\text{Y} - \text{intercept}}{\text{slope}} = \frac{0.0251 \text{ ms}}{0.0014 \text{ s / nF}} = 17.9$ pF as required.

Part II. Determination of geometrical shape of parallel-plates capacitor

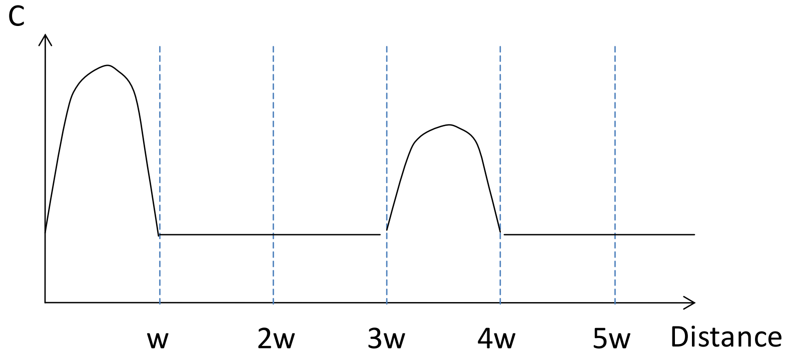
PATTERN I: The expected graph of C versus the position



PATTERN II: The expected graph of C versus the position

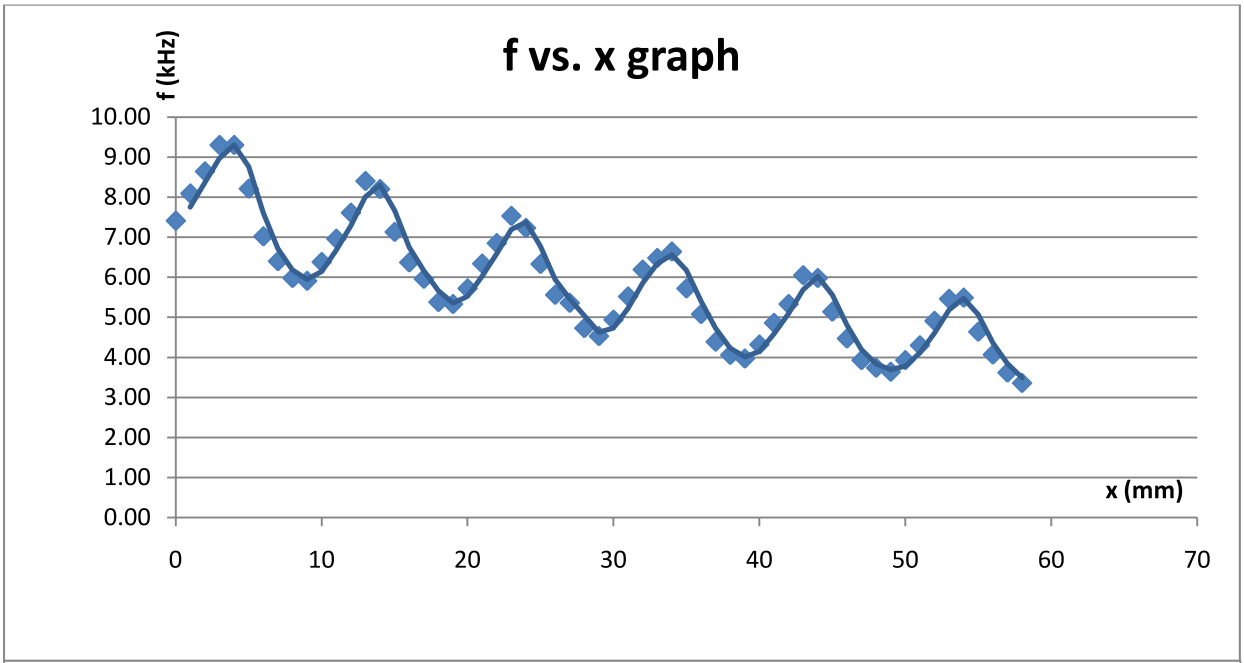


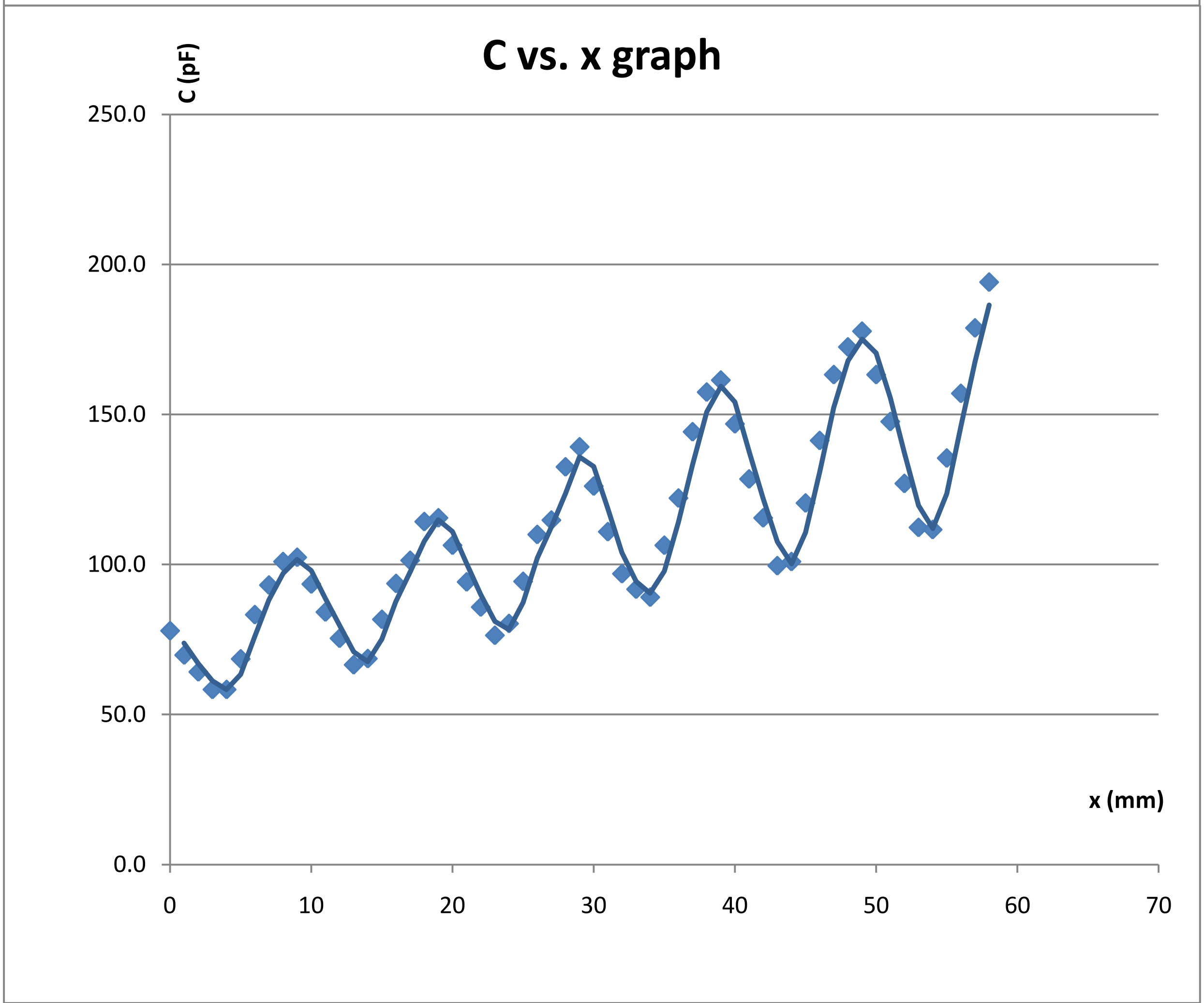
PATTERN III: The expected graph of C versus the position



By measuring f and C versus x (the distance moved between the two plates,) the data and the graphs are shown below.

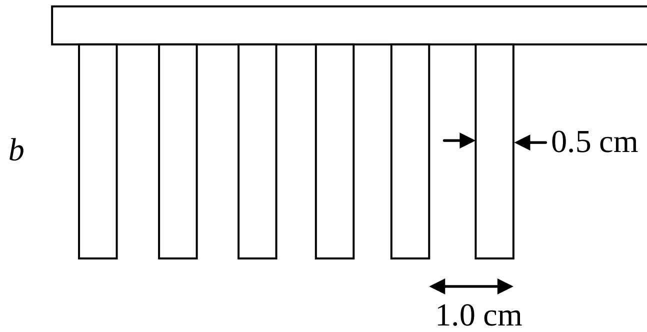
x (mm)	f (kHz)	C (pF)	x (mm)	f (kHz)	C (pF)
0	7.41	77.9	30	4.94	126.1
1	8.09	69.8	31	5.52	110.9
2	8.64	64.2	32	6.19	96.9
3	9.30	58.3	33	6.48	91.7
4	9.30	58.3	34	6.64	89.1
5	8.21	68.5	35	5.72	106.4
6	7.02	83.3	36	5.08	122.1
7	6.40	93.1	37	4.39	144.2
8	5.98	100.9	38	4.06	157.4
9	5.91	102.4	39	3.97	161.4
10	6.38	93.5	40	4.32	146.8
11	6.96	84.1	41	4.86	128.5
12	7.61	75.4	42	5.33	115.5
13	8.40	66.5	43	6.05	99.6
14	8.20	68.6	44	5.98	100.9
15	7.13	81.7	45	5.14	120.5
16	6.37	93.6	46	4.47	141.3
17	5.96	101.3	47	3.93	163.3
18	5.38	114.3	48	3.74	172.5
19	5.33	115.5	49	3.64	177.7
20	5.72	106.4	50	3.93	163.3
21	6.34	94.2	51	4.30	147.6
22	6.85	85.8	52	4.91	127.0
23	7.53	76.4	53	5.46	112.3
24	7.23	80.3	54	5.49	111.6
25	6.33	94.3	55	4.64	135.4
26	5.56	110.0	56	4.07	157.0
27	5.36	114.8	57	3.62	178.8
28	4.73	132.5	58	3.36	194.1
29	4.53	139.2			


f vs. x graph
f (kHz)
10.00
9.00
8.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
0
10
20
30
40
50
60
70
x (mm)


C vs. x graph
C (pF)
250.0
200.0
150.0
100.0
50.0
0.0
0
10
20
30
40
50
60
70
x (mm)

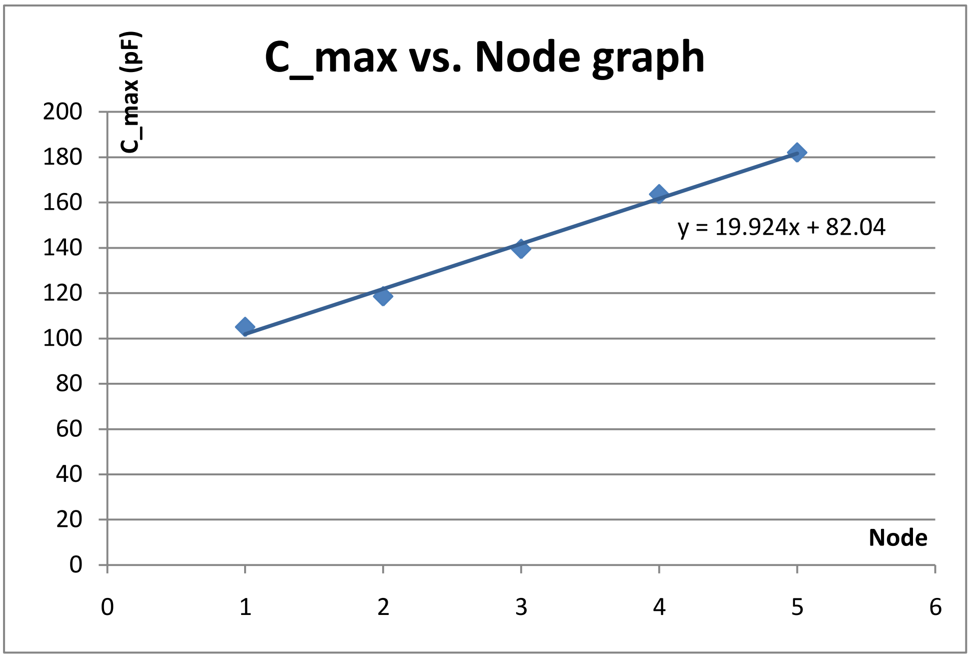
From periodicity of the graph, period = 1.0 cm

Simple possible configuration is:



The peaks of C values obtained from the C vs. x graph are provided in the table below. These maximum C are plotted (on the Y-axis) vs. nodes (on the X-axis.)

node	C_max
1	105.1
2	118.6
3	139.5
4	163.7
5	182.1



This graph is linear of which the slope is the dropped off capacitance $\Delta C = 19.9$ pF/section.

Given that the distance between the plates $d = 0.20$ mm, $K = 1.5$,

$$\Delta C \approx \frac{K\varepsilon_0 A}{d},$$

and $A = 5\times 10^{-3}\,\mathrm{m} \times b\,\mathrm{mm} \times 10^{-3}\ \mathrm{m}^2$

Then, $b\ \text{mm}\ \approx \dfrac{\Delta C\ d}{K\varepsilon_0 \times 10^{-3} \times 5 \times 10^{-3}} \approx 60\ \text{mm}$ if medium between plates is the dielectric of which $K = 1.5$.

Part III. Resolution of digital micrometer

From the given relationship between f and C, $f = \dfrac{\alpha}{C + C_S}$,

$$\Delta f \simeq \left|\frac{df}{dC}\right| \Delta C = \left|\frac{-\alpha}{(C + C_S)^2}\right| \Delta C$$

$$= \frac{f^2}{\alpha} \Delta C$$

$$\Leftrightarrow \qquad \Delta C = \frac{\alpha}{f^2} \Delta f$$

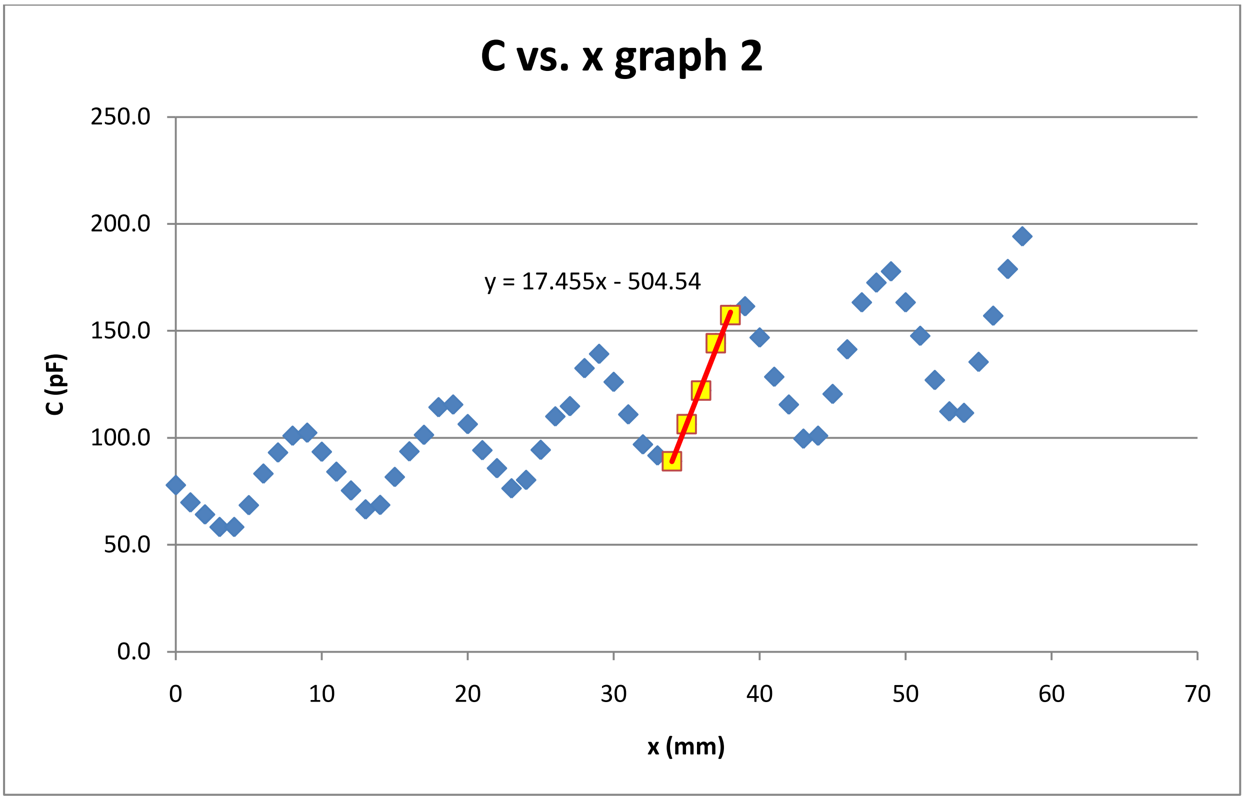
And since C linearly depends on x, $C = mx + \beta \quad \Rightarrow \quad \Delta C = m\Delta x$.

Hence,

$$\Delta x = \frac{\alpha}{mf^2} \Delta f,$$

where Δf is the smallest change of the frequency f which can be detected by the multimeter, x_0 is the operated distance at f = 5 kHz, and m is the gradient of the C vs. x graph at $x = x_0$.

From the f vs. x graph, at f = 5 kHz, The gradient is then measured on the C vs. x graph around this range.



From this graph, $m = 17.5\ \text{pF} / \text{mm} = 1.75 \times 10^{-8} \text{F} / \text{m}$.
Using this value of m, $f = 5\ \text{kHz}, \alpha = 714\ \text{nF/s}$, and $\Delta f = 0.01\ \text{kHz}$,

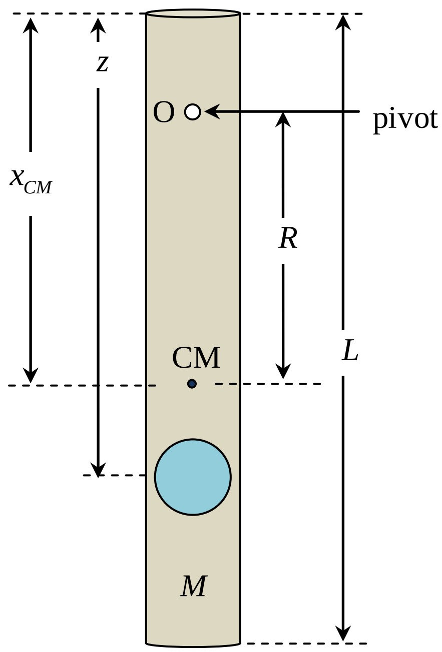
$$\Delta x = \frac{714 \times 10^{-9}}{(1.75 \times 10^{-8})(5 \times 10^{3})^{2}} \times (0.01 \times 10^{3}) = 0.016\,\text{mm}$$

NB. The C vs. x graph is used since C (but not f) is linearly related to x.

Alternative method for finding the resolution
(not strictly correct)

Using the f vs. x graph and the data in the table around $f = 5\ \text{kHz}$, it is found that when f is changed by 1 kHz ($\Delta f = 1\ \text{kHz}$,) x is roughly changed by 1.5 mm ($\Delta x \simeq 1.5\ \text{mm}$.) Hence, when f is changed by $\Delta f = 0.01\ \text{kHz}$ (the smallest detectable of the change,) the distance moved is $\Delta x \simeq 0.015\ \text{mm}$.

Solution: 2 . Mechanical Blackbox: a cylinder with a ball inside



In order to be able to calculate the required values in i, ii, iii, we need to know:

a. the position of the centre of mass of the tubing plus particle (object) which depends on z, m, M
b. the moment of inertia of the above.

The position of the CM may be found by balancing. The I_{CM} can be calculated from the period of oscillation of the tubing plus object.

Analytical steps to select parameters for plotting

I. $$x_{CM} = \frac{mz + M\frac{L}{2}}{m+M} \qquad \text{.......................} (1)$$

L is readily obtainable with a ruler.

x_{CM} is determined by balancing the tubing and object.

II. For small-amplitude oscillation about any point O the period T is given by considering the equation:

$$\left\{(M+m)R^2 + I_{CM}\right\}\ddot{\theta} = -g(M+m)R\sin\theta \approx -g(M+m)R\theta \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

$$T = 2\pi\sqrt{\frac{I_{CM}+(M+m)R^2}{g(M+m)R}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where

$$\begin{aligned} I_{CM} &= \frac{1}{3}M\left(\frac{L}{2}\right)^2 + M\left(x_{CM}-\frac{L}{2}\right)^2 + m\left(z-x_{CM}\right)^2 \\ &= \frac{1}{3}ML^2 + Mx_{CM}^2 - MLx_{CM} + m\left(z-x_{CM}\right)^2 \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Note that

$$T^2\frac{g(M+m)}{4\pi^2} = \frac{I_{CM}}{R} + (M+m)R \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Method (a): (linear graph method)

The equation (5) may be put in the form:

$$T^2R = \left(\frac{4\pi^2}{g}\right)R^2 + \frac{4\pi^2 I_{CM}}{(M+m)g} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Hence the plot of T^2R v.s. R^2 will yield the straight line whose

Slope $\alpha = \dfrac{4\pi^2}{g}$ ……………………… (7)

and y-intercept $\beta = \dfrac{4\pi^2 I_{CM}}{(M+m)g}$ ……………………… (8)

Hence, $I_{CM} = (M+m)\dfrac{\beta}{\alpha}$ ……………………… (9)

The value of g is from equation (7): $g = \dfrac{4\pi^2}{\alpha}$ ……………………… (10)

Method (b): minimum point curve method

The equation (5) implies that T has a minimum value at

$$R = R_{\min} \equiv \sqrt{\frac{I_{CM}}{M+m}} \qquad \text{……………………. (11)}$$

Hence $R_{\min}$ can be obtained from the graph T v.s. R.

And therefore $$I_{CM} = (M+m)R_{\min}^2 \qquad \text{……………………. (12)}$$

This equation (12) together with equation (1) will allow us to calculate the required values z and $\frac{M}{m}$.

At the value $R = R_{\min}$ equation (5) becomes $T_{\min}^2 \frac{g(M+m)}{4\pi^2} = (M+m)R_{\min} + (M+m)R_{\min}$

$$g = \frac{2R_{\min}}{T_{\min}^2} \times 4\pi^2 = \frac{8\pi^2 R_{\min}}{T_{\min}^2} \qquad \text{……………………. (13)}$$

from which g can be calculated.

Results

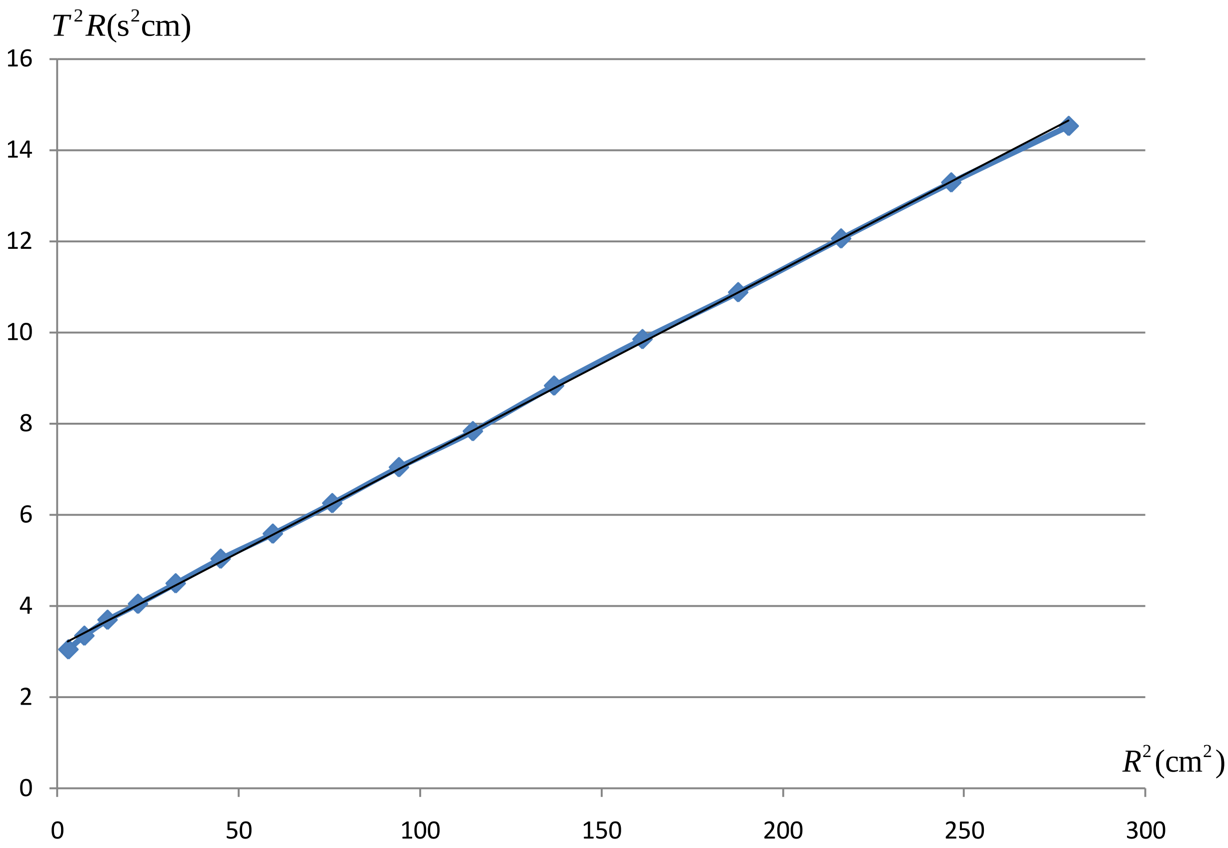
$$L = 30.0 \text{ cm} \pm 0.1 \text{ cm}$$

$$x_{CM} = 17.8 \text{ cm} \pm 0.1 \text{ cm (from top)}$$

$x_{CM} - R$ (cm)	time (s) for 20 cycles			T (s)	R (cm)	R^2 (cm^2)	T^2R (s^{2}cm)
1.1	18.59	18.78	18.59	0.933	16.7	278.9	14.53
2.1	18.44	18.25	18.53	0.920	15.7	246.5	13.29
3.1	18.10	18.09	18.15	0.906	14.7	216.1	12.06
4.1	17.88	17.78	17.81	0.891	13.7	187.7	10.88
5.1	17.69	17.50	17.65	0.881	12.7	161.3	9.85
6.1	17.47	17.38	17.28	0.869	11.7	136.9	8.83
7.1	17.06	17.06	17.22	0.856	10.7	114.5	7.83
8.1	17.06	17.00	17.06	0.852	9.7	94.1	7.04
9.1	16.97	16.91	16.96	0.847	8.7	75.7	6.25
10.1	17.00	17.03	17.06	0.852	7.7	59.3	5.58
11.1	17.22	17.37	17.38	0.866	6.7	44.9	5.03
12.1	17.78	17.72	17.75	0.888	5.7	32.5	4.49
13.1	18.57	18.59	18.47	0.927	4.7	22.1	4.04
14.1	19.78	19.90	19.75	0.991	3.7	13.7	3.69
15.1	11.16	11.13	11.13	1.114	2.7	7.3	3.34
16.1	13.25	13.40	13.50	1.338	1.7	2.9	3.04

Notes: at $x_{CM} - R = 15.1, 16.1$ cm, times for 10 cycles.

Method (a)



Calculation from straight line graph: slope $\alpha = 0.04108 \pm 0.0007$ s^2/cm, y-intercept $\beta = 3.10 \pm 0.05$ s^{2}cm

$$g = \frac{4\pi^2}{\alpha} \text{ giving } g = (961 \pm 20) \text{ cm/s}^2$$

$$\frac{\beta}{\alpha} = \frac{3.10}{0.04108} = 75.46 \text{ cm}^2 \left(\pm 2.5\,\text{cm}^2\right)$$

$$I_{CM} = (M+m)\frac{\beta}{\alpha} = (75.46)(M+m)$$

From equation (4): $$I_{CM} = \frac{1}{3}M\left(\frac{L}{2}\right)^2 + M\left(x_{CM} - \frac{L}{2}\right)^2 + m\left(z - x_{CM}\right)^2$$

Then $$(75.46)(M+m) = 75.0M + 7.84M + m(z-17.8)^2$$

$$-7.38\frac{M}{m} + 75.46 = (z-17.8)^2 \qquad \text{……………………. (14)}$$

The centre of mass position gives:

$$17.8(M+m) = 15.0M + mz$$

$$\frac{M}{m} = \frac{z-17.8}{2.8} \qquad \text{……………………. (15)}$$

From equations (14) and (15):

$$-\frac{7.38}{2.8}(z-17.8) + 75.46 = (z-17.8)^2$$

$$(z-17.8) = 7.47$$

And $$z = 25.27 = 25.3 \pm 0.1 \text{ cm}$$

$$\frac{M}{m} = 2.68 = 2.7$$

Error Estimation

Find error for g:

From (10), $$g = \frac{4\pi^2}{\alpha}$$

$$\Delta g = \frac{\Delta\alpha}{\alpha}g = 16.3 \text{ cm/s}^2 \approx 20 \text{ cm/s}^2$$

i) Find error for z:

First, find error for $r = \frac{\beta}{\alpha} = \frac{3.10}{0.04108} = 75.46 \text{ cm}^2$.

$$\Delta r = \left(\frac{\Delta\alpha}{\alpha} + \frac{\Delta\beta}{\beta}\right)r = 2.5 \text{ cm}^2$$

Since error from r contributes most ($\frac{\Delta r}{r} \sim 0.03$ while $\frac{\Delta L}{L}, \frac{\Delta x_{cm}}{x_{cm}} \sim 0.005$), we estimate error propagation from r only to simplify the analysis by substituting the min and max values into equation (4).

Now, we use $r_{\max} = r + \Delta r = 75.46 + 2.5 = 77.96$. The corresponding quadratic equation is $(z-17.8)^2 + 1.743(z-17.8) - 77.96 = 0$ The corresponding solution is $(z-17.8)_{\max} = 7.55$ cm

If we use $r_{\min} = r - \Delta r = 75.46 - 2.5 = 72.96$, the corresponding quadratic equation is

$$\left(z-17.8\right)^2 + 3.529\left(z-17.8\right) - 72.96 = 0$$

The corresponding solution is $(z-17.8)_{\min} = 6.96$ cm

So $\Delta(z-17.8) = \dfrac{7.55-6.96}{2} = 0.3$ cm

Note that $\dfrac{\Delta(z-17.8)}{z-17.8} \sim 0.04$. So, we still ignore the error propagation due to $\Delta L, \Delta x_{cm}$

The error Δz can be estimated from $\Delta z \approx \Delta(z-17.8) = 0.3$ cm

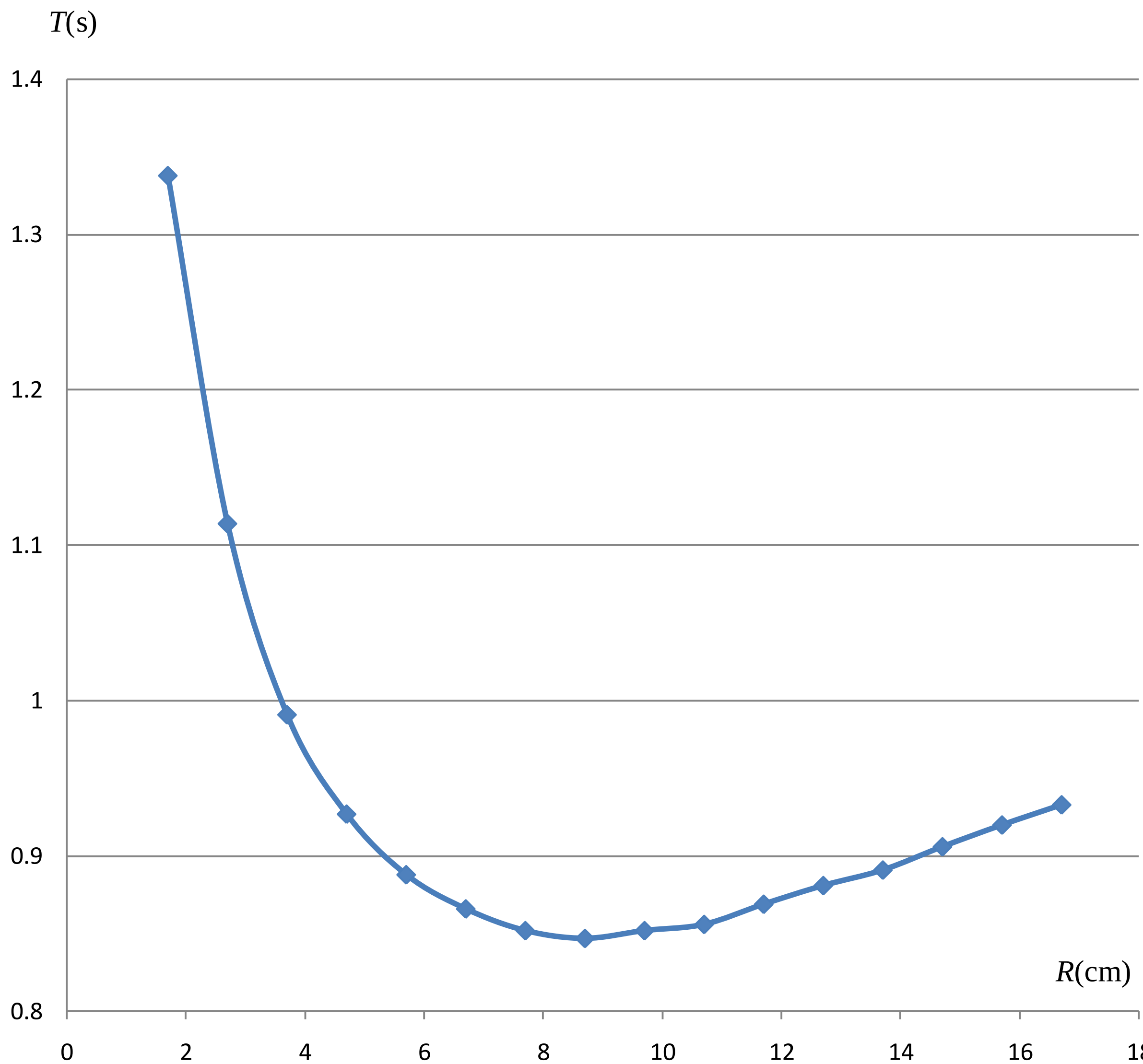
ii) Find error for $\dfrac{M}{m}$:

We know that $\dfrac{M}{m} = \dfrac{z-17.8}{2.8}$

$$\Delta\left(\frac{M}{m}\right) = \frac{\Delta(z-17.8)}{2.8} = 0.11$$

Method (b)

Calculation from *T*-*R* plot:



Using the minimum position: $T = T_{\min}$ at $I_{CM} = (M+m)R_{\min}^2$ and $g = \dfrac{8\pi^2 R_{\min}}{T_{\min}^2}$

From graph: $R_{\min} = 8.9 \pm 0.2$ cm and $T_{\min} = 0.846 \pm 0.005$ s

$$\therefore \quad g = 982 \pm 40 \text{ cm/s}^2$$

$$I_{CM} = (M+m)(8.9)^2 = (79.21)(M+m) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)$$

From equations (14), (15), (16):

$$(79.21)(M+m) = 75.0M + 7.84M + m(z-17.8)^2$$

$$-3.63M + 79.21m = m(z-17.8)^2$$

$$(x-17.8)^2 + \frac{3.63}{2.8}(x-17.8) - 79.21 = 0$$

$$(z-17.8) = 8.28$$

And $z = 26.08 = 26.1 \pm 0.7$ cm

$$\frac{M}{m} = 2.95 = 3.0 \pm 0.3$$

<u>Error estimation</u>

i) Find error for g:

Using the minimum position: $g = \dfrac{8\pi^2 R_{\min}}{T_{\min}^2}$, we have

$$\Delta g = \left(\frac{\Delta R_{\min}}{R_{\min}} + 2\frac{\Delta T_{\min}}{T_{\min}}\right) g = 34 \approx 30 \text{ cm/s}^2$$

ii) Find error for z:

First, find error for $r = R_{\min}^2 = 79.21 \text{ cm}^2$.

$$\Delta r = 2R_{\min}\Delta R_{\min} = 3.56 \text{ cm}^2$$

This r is equivalent to r in part 1. So, one can follow the same error analysis.
As a result, we have

$$z = 26.08 \approx 26.1 \text{ cm}$$
$$\Delta z = 0.8 \text{ cm}$$

i) Find error for $\dfrac{M}{m}$:

Following the same analysis as in part I, we found that

$$\frac{M}{m} = 2.96\,;\ \Delta\left(\frac{M}{m}\right) = 0.15$$

NOTE: This minimum curve method is not as accurate as the usual straight line graph.